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ANALYSIS OF BRACED FRAMES

Kurt H. Gerstle,[1] J.M. ASCE
(Proc. Paper 1560)

ABSTRACT

A method of analysis for knee-braced frames is presented which is analogous to Prof. Cross' Moment Distribution for rigid-jointed frames. Values of fixed support forces, distribution factors, and carry-over factors are given, and the procedure is explained by means of an example problem.

INTRODUCTION

Braced frames of the type shown in Fig. 1 are frequently used in industrial and other structures where stiffness and ease of erection are required, and where there is no objection to the knee braces. Even though this kind of structure is closely related to rigid-jointed frames, no method of analysis comparable in convenience to moment distribution has been devised for it. The designer confronted with a braced frame will either use the method of virtual work which becomes tedious for highly indeterminate frames, or resort to some approximate method the degree of exactness of which might be subject to doubt.

In this paper a method of analysis is presented which is based on principles first stated by Prof. Hardy Cross.[2] Whereas in Prof. Cross' moment distribution procedure the unbalanced moments at a joint are distributed to the members in proportion to their rotational stiffness, here the unbalanced forces at a brace are distributed to the members depending on their deflection stiffness. The familiar concepts of locking a portion of the structure, distribution, carry-over, etc., are retained in the proposed method, so that the

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1. Asst. Prof. of Civ. Eng., Univ. of Colorado, Boulder, Colo.
2. Cross, Hardy, "Analysis of Continuous Frames by Distributing Fixed-End Moments", ASCE Trans., Vol. 96, 1932.

analyst familiar with moment distribution should have little difficulty with the approach presented here.

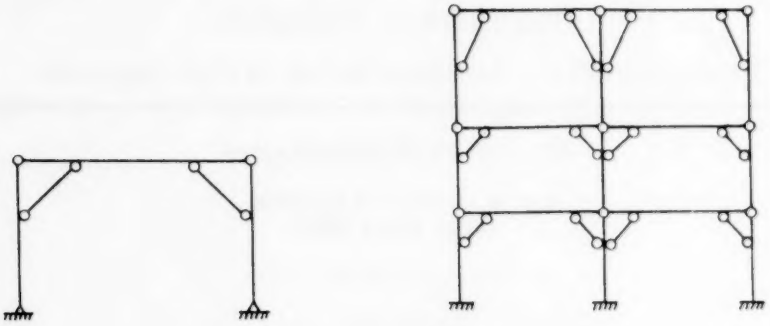


Fig.1 Braced Frames

Assumptions

The usual assumptions of structural analysis regarding elastic behavior, small deflections, etc., underlie the proposed method. Axial deformations are neglected. This last simplification applied to the brackets will lead to a certain amount of error, which, however, does not exceed the allowable tolerances of ordinary design practice for usual structures.

Theory

The following steps outline the general procedure followed:

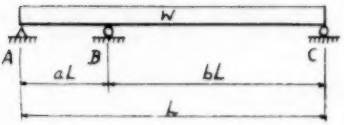
1. Assume all attachment points of braces to members to be held against deflection, and compute the holding forces required to accomplish this. These forces will be called "Fixed Support Forces" (F.S.F.). The calculation of these forces is discussed below.
2. The unbalanced holding forces for each set of braces are distributed to the adjacent members according to rules presented in the Section "Distribution Factors".
3. The effect of the forces allocated to each member in Step 2 upon the reactions at the far end of the member is calculated by means of the carry-over factor.
4. The forces or moments computed in Step 3 are new fixed support forces or fixed end moments, thereby initiating a new cycle of distribution.
5. The process is repeated till the required degree of accuracy is obtained.

Sign Convention

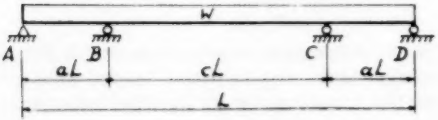
A force exerted by a brace tending to produce clockwise rotation of the joint which is stiffened by that brace will be called positive. This is in keeping with the usual moment distribution convention, and will enable a structure containing both knee-braced and rigid joints to be analyzed by the proposed method. In this case, in the outline above, not only fixed support forces, but also fixed end moments should be considered.

Fixed Support Forces

The holding forces for any loading condition can be calculated as the reactions of continuous beams. For two cases of frequent occurrence, formulas are given for fixed support forces.



$$R_B = \left[\frac{1}{2} + \frac{1}{8ab}(b^3 + a^3)\right] WL.$$

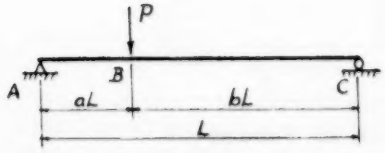


$$R_B = \left[\frac{5}{8}a + \frac{1}{2}c + \frac{c}{4a(2c+1)}\left(c^2 - \frac{3}{2}a^2\right)\right] WL.$$

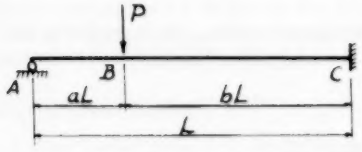
Fig.2 Fixed Support Forces

Stiffness Factors

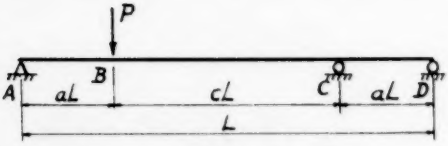
Stiffnesses required by this method measure resistance to deflection at a point, and are defined as $\frac{P}{\Delta}$, where P is the force required to impart a deflection Δ at that point. For usual cases, stiffness values are presented here:



$$K_B = \frac{P_B}{\Delta_B} = \frac{3EI}{L^3}\left(\frac{1}{a^2 b^2}\right),$$



$$K_B = \frac{P_B}{\Delta_B} = \frac{3EI}{L^3}\left[\frac{4}{a^2 b^3 (3+a)}\right],$$



$$K_B = \frac{P_B}{\Delta_B} = \frac{3EI}{L^3}\left[\frac{4(1-a)^2}{a^2c^2(1+2c)}\right].$$

Fig.3 Stiffness Factors

Distribution Factors

A part of a braced frame subjected to unbalanced forces is shown in Fig. 4. Three members 1, 2, and 3 are pinned at a joint, and stiffened by two knee braces attached at angles α and β with the center member 2. The deflection stiffnesses of members 1, 2, and 3 at the points of bracket attachment are denoted by K_1, K_2, and K_3.

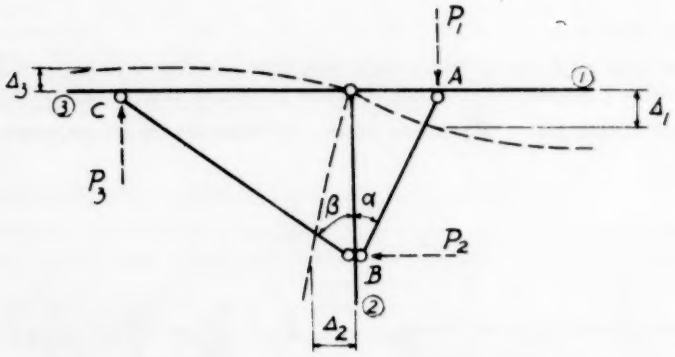


Fig.4 Distribution of Forces

If for any reason, such as application of unbalanced forces, the members deform as shown, then the forces resisted by the members are

$$P_1 = K_1 \Delta_1$$

$$P_2 = K_2 \Delta_2 = K_2 \frac{\Delta_1}{\text{Tan}\,\alpha} \qquad (1)$$

$$P_3 = K_3 \Delta_3 = K_3 \frac{\Delta_1 \text{ Tan}\,\beta}{\text{Tan}\,\alpha}.$$

Thus, the three resisting forces are in a definite ratio to each other, depending only on the properties of the structure.

It should be noted that the sum of the forces P_1, P_2, and P_3 need not necessarily be equal to the unbalanced load to be distributed. Rather, the relationship is given by the laws of statics, which, for an unbalanced force P_A applied at Point A, require that

$$P_A = P_1 + \frac{P_2}{\text{Tan}\alpha} + P_3 \frac{\text{Tan}\beta}{\text{Tan}\alpha} \qquad (2)$$

Combining Eqs. (1) and (2), it is seen that, due to P_A,

$$P_1 = \frac{K_1 \text{Tan}\alpha}{K_1 \text{Tan}^2\alpha + K_2 + K_3 \text{Tan}^2\beta} \text{Tan}\alpha \; P_A$$

$$P_2 = \frac{K_2}{K_1 \text{Tan}^2\alpha + K_2 + K_3 \text{Tan}^2\beta} \text{Tan}\alpha \; P_A$$

$$P_3 = \frac{K_3 \text{Tan}\beta}{K_1 \text{Tan}^2\alpha + K_2 + K_3 \text{Tan}^2\beta} \text{Tan}\alpha \; P_A \; .$$

Static relationships such as Eq. (2) may also be written for cases where the unbalanced force is applied at Points B or C. These expressions are then combined with Eq. (1) to arrive at the following values for distribution factors: Due to P_B,

$$P_1 = \frac{K_1 \text{Tan}\alpha}{K_1 \text{Tan}^2\alpha + K_2 + K_3 \text{Tan}^2\beta} \; P_B$$

$$P_2 = \frac{K_2}{K_1 \text{Tan}^2\alpha + K_2 + K_3 \text{Tan}^2\beta} \; P_B$$

$$P_3 = \frac{K_3 \text{Tan}\beta}{K_1 \text{Tan}^2\alpha + K_2 + K_3 \text{Tan}^2\beta} \; P_B$$

Due to P_C,

$$P_1 = \frac{K_1 \text{Tan}\alpha}{K_1 \text{Tan}^2\alpha + K_2 + K_3 \text{Tan}^2\beta} \text{Tan}\beta \; P_C$$

$$P_2 = \frac{K_2}{K_1 \text{Tan}^2\alpha + K_2 + K_3 \text{Tan}^2\beta} \text{Tan}\beta \; P_C$$

$$P_3 = \frac{K_3 \text{Tan}\beta}{K_1 \text{Tan}^2\alpha + K_2 + K_3 \text{Tan}^2\beta} \text{Tan}\beta \; P_C$$

It is seen that in the general case, the distribution factors vary according to the point of application of the force to be distributed, and the values of the balancing forces do not add up to the value of the unbalanced force. However, in the special case in which the braces are attached at 45° with the members, Tan α = Tan β = 1, and the distribution factors reduce to the familiar relations of moment distribution.

Carry-Over Factors

The forces produced in members during the distribution will in turn induce reactive forces and moments at support points. Carry-over factors are defined as the ratio of the value of the induced force or moment to the value of the force which causes it. Formulas for carry-over factors are given for several common cases:

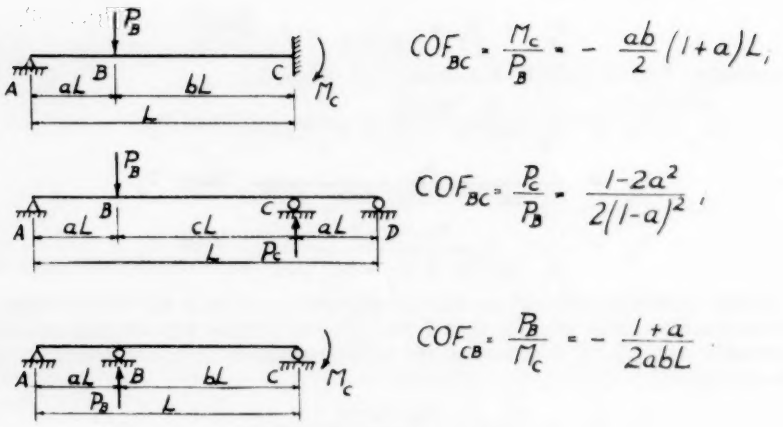


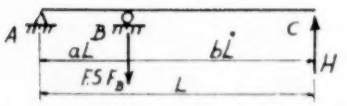
Fig.5 Carry-over Factors

Side Sway

The analysis of braced structures subject to joint translations also follows closely the precepts commonly employed in moment distribution. The steps may be enumerated:

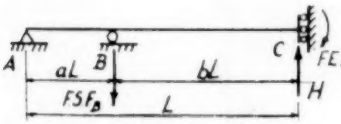
1. The structure is assumed held against sway, and analyzed in this condition. An unbalanced shear results.
2. The structure is displaced an arbitrary amount Δ, resulting in fixed support forces and fixed end moments. These unbalanced reactions are due to the shear in each member, the shear being proportional to the rigidity of the member.
3. The unbalanced F.S.F.'s and F.E.M.'s due to Step 2 are distributed.
4. The results of Step 3 are multiplied by a ratio so as to annul the unbalanced shears of Step 1.

Values of shear rigidities, fixed support forces, and fixed end moments for two common cases are given:



$$\text{Rigidity} = \frac{H}{\Delta_C} = \frac{3EI}{L^3}\,\frac{1}{b^2},$$

$$FSF_B = \frac{H}{a};$$



$$\text{Rigidity} = \frac{H}{\Delta_C} = \frac{3EI}{L^3}\,\frac{4(1+2b)}{b(3+a)},$$

$$FSF_B = \frac{2+2b-b^2}{2+2b-4b^2}\,H,$$

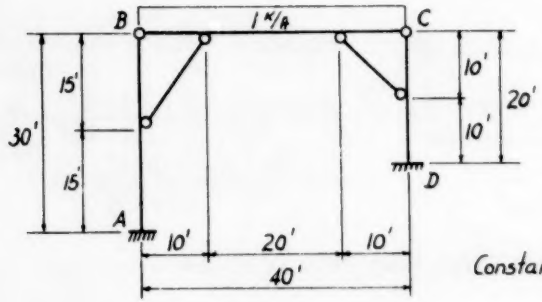
$$FEM_C = \frac{(2+b)bL}{2(1+2b)}\,H.$$

Fig.6 Side Sway Relationships

CONCLUSIONS

A method for analyzing braced frames has been presented which might in some cases result in a saving of time and effort over established procedures. The approach can be further refined by utilizing successive shear corrections and other methods found useful in moment distribution. Other modifications might concern themselves with the effects of axial deformation of the braces and of continuity of the columns.

EXAMPLE PROBLEM



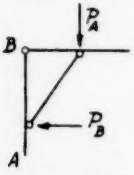
a, Neglecting Sway :

F.S.F. = 17.81 ^{K}, (Fig. 2)

K_{BA} = 1.354, K_{BC} = 1.125, K_{CD} = 4.571, (Fig. 3)

Distr. Factors (Eq. 3)

Joint B :



Due to P_A : D.F. BC = .270, BA = .487,

Due to P_B : BC = .405, BA = .730,

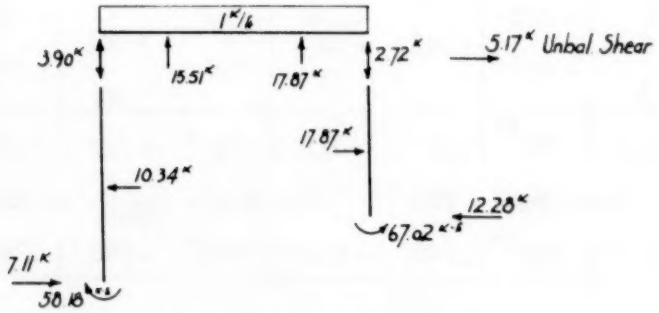
Joint C : D.F CB = .198, CD = .802;

C.O.F.'s (Fig. 5) :

BA = − 5.625 $^{K\text{-}ft/K}$, BC = CB = + 778 $^{K/K}$,

CD = − 3.75 $^{K\text{-}ft/K}$.

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
	Mom	Force		Force		Mom
D.F. Vert Ld.		.487	.270	.198	.802	
D.F. Hor. Ld.		.730	.405			
C.O.F.		-.5625	+.778	+.778	-3.75	
F.S.F.			-17.81	+17.81		
B		+8.67	+4.81	-3.53	-14.28	
C	-48.77		-2.75	+3.74		+53.55
B		+1.34	+.74	-.74	-3.00	
C	-7.54		-.58	+.58		+11.25
B		+.28	+.16	-.11	-.47	
C	-1.58		-.09	+.12		+1.76
B		+.04	+.02	-.02	-.10	
C	-.23		-.02	+.02		+.38
B		+.01	+.01	-	-.02	
C	-.06		-	-		+.08
B		-	-	-	-	
M or F	-58.18 K-ft	+10.34 K	-15.51 K	+17.87 K	-17.87 K	+67.02 K-ft



Forces - Side Sway Neglected

b, Correction for Side Sway

Relative Rigidities (Fig 6) : $H_{AB} \sim .229$,

$H_{CD} \sim .771$,

F.S.F. due to H: (Fig. 6) : AB ~ .229^K, CD ~ .771^K,

F.E.M. due to H: (From Fig 6): $FEM = \frac{5}{22} L \cdot F.S.F$

$\therefore$ AB = −1.561$^{K\cdot ft}$, DC = −3.505$^{K\cdot ft}$

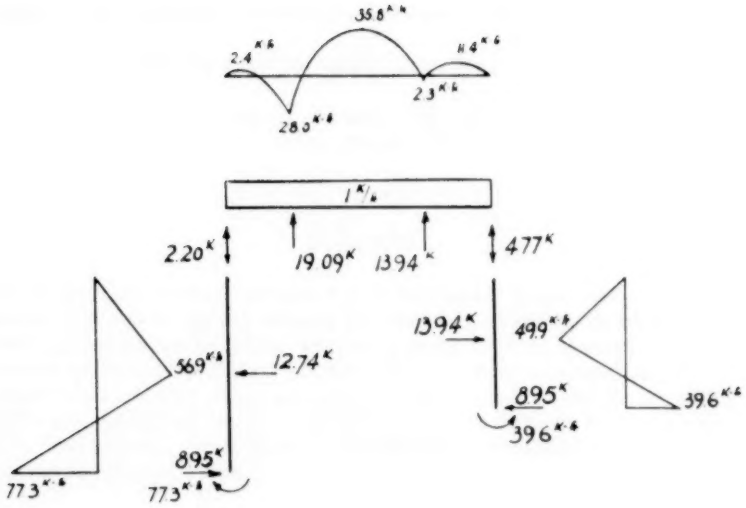
Joint	A	B		C		D
Member	AB	BA	BC	BC	CB	DC
	Mom.	Force		Force		Mom.
D.F. Vert Ld		.487	.270	.198	.802	
D.F. Hor Ld		.730	.465			
C.O.F.		−.5625	+.778	+.778	−3.750	
F.S.F. & F.E.M.	−1.561	+.229			+.771	−3.505
B		−.167	−.093	−.153	−.618	
C	+.939	−	−.119	−.072	−	+2.318
B		+.058	+.032	+.014	+.058	
C	−.326	−	+.011	+.025	−	−.218
B		−.005	−.003	−.005	−.020	
C	+.028	−	−.004	−.002	−	+.075
B		+.002	+.001	−	+.002	
C	−.011		−	+.001	−	−.008
B					−.001	
M or F	−.931$^{K\cdot ft}$	+.117^K	−.175^K	−.192^K	+.192^K	−1.338$^{K\cdot ft}$

Unbal. Shear = .253^K, Mult Factor = $\frac{5.17^K}{.253^K}$ = 20.483

M or F	−19.07$^{K\cdot ft}$	+2.40^K	−3.58^K	−3.95^K	+3.95^K	−27.41$^{K\cdot ft}$

c, Superposition of Results a, & b.

M or F	−77.25$^{K\cdot ft}$	+12.74^K	−19.09^K	+13.94^K	−13.94^K	+39.61$^{K\cdot ft}$



Final Forces and Moments.

Paper 1561 March, 1958 ST 2

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LATERAL BRACING OF COLUMNS AND BEAMS

George Winter,[1] M. ASCE
(Proc. Paper 1561)

SYNOPSIS

There are many situations where it is necessary to determine the characteristics required of lateral bracing in order to counteract buckling of columns or beams, or to decide whether a given bracing system is adequate to provide the required lateral support. In this paper a simple and elementary method is developed which permits one to calculate lower limits of the strength and rigidity of lateral support in order to provide "full bracing" to columns and beams. "Full bracing" is defined as equivalent in effectiveness to immovable lateral support.

INTRODUCTION

There are two kinds of structural bracing: (a) that provided to resist "secondary" loads (e.g. windbracing) and (b) that provided to increase the strength of structural members by preventing them from deforming in their weakest direction. The present paper refers only to this latter kind of bracing. Here, again, there are two different types of problems: (i) Bracing applied to counteract stable but detrimental types of deformation. As examples one might mention beams of channel or Z-shape, loaded in the plane of the web; such beams unless properly braced, twist and deflect sideways with consequent loss of strength (contributions to this bracing problem have been made in Refs. 1 and 2). (ii) Bracing applied to prevent buckling in the weakest direction, and thereby to increase the usable strength of the member. It is this last kind, bracing against buckling, which is the subject of the present investigation.

Note: Discussion open until August 1, 1958. A postponement of this closing date can be obtained by writing to the ASCE Manager of Technical Publications. Paper 1561 is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 84, No. ST 2, March, 1958.

1. Prof. and Head, Dept. of Structural Eng., Cornell Univ., Ithaca, N. Y.

For the first-mentioned cases, (a) and (b,i) above, it is only necessary to provide bracing of sufficient strength; given the loads there is no basic difficulty in computing the internal forces in the bracing system and to dimension it accordingly. In contrast, for bracing against buckling to be effective it must possess not only the requisite strength but also a definite minimum rigidity; and, as will be seen, neither the requisite strength nor the necessary rigidity can be computed uniquely except on the basis of assumed imperfections of shape and/or loading of the member to be braced.

To mention a few frequent problems of such bracing: To prevent buckling of the compression chord of a roof truss out of its plane, is the strength and rigidity of the roof proper sufficient (steel roof deck, or other planking adequately connected to form a diaphragm) or must special bracing members be provided? For a laced wall column in a mill building, where the outer leg is adequately braced in the plane of the wall, how must horizontal braces be positioned and dimensioned to prevent buckling parallel to the wall of the free-standing inner leg? In order for a beam or girder to be "fully braced", must its compression chord be embedded in a concrete floor slab? If not, what are the required minimum strength and rigidity of the floor system to provide full bracing? Many more such cases could be enumerated, but these few will suffice to illustrate the problem of bracing against buckling.

Characteristics of Effective Bracing

Bracing of this type is of considerable consequence because very large increases in carrying capacity can be achieved by very light, i.e., inexpensive, bracing. This will be illustrated by selected results from a series of tests which were reported previously in detail.[3]

Specimens consisted of two channels cold-formed of 16 ga. sheet steel and spot-welded back to back to form an I-shape 4 in. deep and 2 in. wide. They were tested as columns 12'-3" long between knife edges. While one of the columns was tested unbraced, for the other identical specimens intermediate bracing was provided at equidistant points. This bracing, at each point, consisted of two strips of thin, 10-ply cardboard, 3/4 in. wide and 2 in. to 15 in. long whose combined tensile strength was 103 lb. per pair. The left part of Fig. 1 shows one of the studs under test, with four pairs of strips in place, the right part a portion of a failed specimen with two pairs of broken cardboard strips. (These photos happen to show tests where the strips were 1.5 in. wide.) Table 1 gives a representative selection of test results.

These results illustrate, qualitatively, a number of important points:

(a) In these tests the usable strength of the columns could be increased by up to fifteen times by using bracing as weak as that provided by the described cardboard strips.

(b) In those of the tests where column failure was accompanied by fracture of the bracing strips, the strength of one brace was of the order of only one per cent of the strength of the column.

(c) The efficiency of bracing depends not only on its strength but also on its rigidity. In fact, the strength per brace was the same in all these tests, 103 lb. However, since the longer strips possess proportionately larger extensibility, the shorter the strips the greater the rigidity of the bracing. Comparison of the paired tests 8, 9; 15, 16; and 20, 22 shows, in each case, that the greater rigidity produced the larger

Table 1

Tests on Cardboard-Braced Light-Gage Steel Columns

Test No.	Cardboard Strips number	length, (in.)	Meas. Ultimate Load, (lbs.)	Brace Strength, % of Column Strength	Strips broke
0	0	-	1098	-	-
4	1 pair	15	4650	2.2	no *)
8	2 pairs	15	8600	1.2	yes
9	2 pairs	4	9900	1.0	no
15	3 pairs	15	10,450	1.0	yes
16	3 pairs	2	15,200	0.7	yes
20	4 pairs	15	11,600	0.9	yes
22	4 pairs	2	17,000	0.6	no

*) In tests No. 4, 9, and 22 failure occurred by buckling of the columns over the free lengths between braces, with bracing strips remaining intact.

column capacity. In addition, tests 8, 9 and 20, 22 illustrate that, other things being equal, the greater the rigidity of the bracing the smaller the strength required of it to produce a given column capacity. Indeed, in each of these pairs the softer (15") bracing broke while the more rigid (shorter) bracing not only produced a larger column load but, moreover, did not break at that load. Although the strips did break in both tests 15, 16 it is seen that here, too, of two bracings of the same strength the stiffer one produced the higher column load before failure occurred by fracture of the bracing.

It is possible, in mathematically rather elaborate ways, to investigate this situation accurately by an analysis of elastically supported columns with initial imperfections. (On the basis of somewhat approximate assumptions a few special cases of this kind have been analyzed in Refs. 3 and 4.) Unfortunately such analyses lead to results far too complex for direct practical application. However, the small magnitudes of both rigidity and strength of the bracing which are sufficient to produce extremely large effects, as illustrated in Table 1, suggest that it is not necessary to compute these two characteristics with great accuracy. Instead, it is sufficient, practically, to devise a simple method which permits:

(1) To calculate a safe lower limit (rather than an exact value) of the necessary rigidity of the bracing such that the strength of the braced member will attain its maximum possible value, and
(2) For a bracing of rigidity equal to or larger than so calculated, to determine a safe, lower limit (rather than an exact value) of the strength required of such bracing.

The simple analysis which follows, attempts to provide such a method.

Point-Braced Columns

Analysis

As an acceptable approximation, the performance of practical means of bracing can be regarded as elastic. Consequently, for "ideal" columns (straight, concentrically loaded, etc.) the well-established theory of columns on elastic supports permits one for a given column to determine the relation between support rigidity (spring constant) and column load, at least for simple cases of equidistant supports. (Refs. 5, 3, and 6). These investigations give no information on required strength of support, since, for such "ideal" conditions, the reaction at the support is zero up to the moment of buckling, and indeterminate when buckling occurs.

However, for most design situations the question is not that of determining the buckling load for a given rigidity of support, the former more or less steadily increasing with the latter up to a certain limit. The question is, much simpler, to determine that minimum rigidity which will make the actually elastic bracing equivalent in effect to an unyielding support. Such bracing will evidently produce the maximum column load for the given location of brace or braces and will be called "full bracing". From the data of Table 1 it is evident that relatively low rigidities and strengths are required to provide such full bracing. Consequently, any bracing of lesser effectiveness will hardly be less costly, and is therefore uneconomical. For this reason, and in accord with points (1) and (2) of the preceding section, it is the main aim of the present investigation to calculate the required characteristics of such "full bracing".

If an "ideal column" with hinged ends is furnished with an unyielding, knife-edge support at midlength (Fig. 2), it buckles in two half sine-waves. At the intermediate support $y'' = -M/EI = 0$. Consequently, if a real or fictitious hinge were introduced at the support in the continuous column, as shown in Fig. 2, nothing would be changed. In particular, the column, with or without such hinge, buckles at the Euler-Shanley load

$$P_e = \frac{\pi^2 E I}{L^2} \qquad (1)$$

where in the elastic domain E is Young's modulus and in the plastic domain it is the tangent modulus.

If, now, this ideal column is braced by an elastic support (Fig. 3) and if this support is rigid enough to provide "full bracing", the column buckles in exactly the same manner as for unyielding support, and again a fictitious hinge can be introduced at the support (Fig. 3).

To obtain information for practical design of bracing it is now necessary to investigate 'real' columns. This is so because, as already mentioned, the theory of ideal columns would indicate that supports of infinitesimally small strength are sufficient, which is evidently not a permissible idealization and is at variance with the information of Table 1. Also, as will become apparent, the minimum rigidities calculated for full bracing of ideal columns are not sufficient to achieve full bracing of real, i.e. imperfect columns. To simplify analysis it will be assumed that actual column imperfections (eccentricity, curvature, etc.) can be represented by an equivalent, initial crookedness or "bow", whose amplitude d_o is small as compared to L.

Such a column, with single elastic support, is shown in Fig. 4. As the column is loaded, the elastic brace with spring constant k deflects together with the column as is shown in broken lines, with a consequent support reaction F = k.d. If "full bracing" is provided, final failure occurs at a load equal (or almost equal) to P_e at which load the column snaps into the two-half-wave mode shown solidly. In this case, as before, a fictitious hinge can be introduced at the support with negligible error.

Taking moments about this hinge:

$$M = \frac{F\,L}{2} - P_e\,(d_o + d) = 0$$

Introducing F = k.d one obtains the spring constant required to produce full bracing:

$$k_{req} = \frac{2\,P_e}{L}\,(d_o/d + 1) \qquad (2)$$

For an "ideal column" $d_o = 0$ and one has

$$k_{id} = \frac{2\,P_e}{L} = \frac{2\pi^2 E\,I}{L^3} \qquad (3)$$

This value is identical with that given by the exact theory of elastically supported columns, for that amount of rigidity which is required to make an elastic support as effective as an unyielding support (see Appendix). It will be observed that the rigidity required to produce full bracing in imperfect columns (Eq. 2) exceeds that required for the ideal column (Eq. 3), the more so the larger the imperfection d_o.

If one knows or estimates the initial imperfection d_o (being guided, for instance, by specified tolerances for crookedness) and if one stipulates the maximum permissible deflection d immediately preceding failure, (for example d = L/500, conservatively), one can then calculate the required rigidity k_{req}.

The required strength of bracing S is equal to the reaction F at, or immediately preceding, buckling, i.e.

$$S_{req} = k_{req}d = \frac{2\,P_e}{L}\,(d_o + d) = k_{id}\,(d_o + d) \qquad (4)$$

It can now be said that "full bracing" has been provided if

$$k_{act} \geq k_{req} \quad \text{and} \quad S_{act} \geq S_{req} \qquad (5)$$

Next, columns with two and three equal and equidistant supports will be briefly considered, whereupon certain generalizations will become apparent.

A column with two equidistant supports, when loaded with P_e so that fictitious hinges can be assumed at these supports, could buckle in one of the two modes shown on Fig. 5. To decide which of these modes is more unfavorable, i.e. requires the larger support rigidity k, the case of the "ideal" column will be investigated first. From Fig. 5a

$F.L = P_e.d = 0$ which, with $F = k_{id}d$ gives

$$k_{id} = \frac{P_e}{L} \qquad (a)$$

On the other hand, from Fig. 5b:

$$\frac{F}{3} L - P_e \cdot d = 0 \text{ which, with } F = k_{id} d \text{ gives}$$

$$k_{id} = \frac{3 P_e}{L} \tag{6}$$

It is seen that k_{id} for the mode of Fig. 5b exceeds considerably that for Fig. 5a. This means that if actually k had only the value given by Eq. (a), the column could not reach the value P_e and buckle in the mode of Fig. 5a because at some lower load it would snap into and buckle in the mode of Fig. 5b. It is, consequently, the latter mode which governs and determines the magnitude of k_{id}. That this is so can also be shown by simple energy considerations or from the complete theory of elastically supported columns summarized in the Appendix. (The method of fictitious hinges for finding k_{id} for "full bracing" of ideal columns has been used by Fr. Bleich, Ref. 7).

To obtain values for k_{req} and S_{req} it is now necessary, as in the previous case, to assume an initially curved shape. Since the ideal column buckles according to Fig. 5b, it is evidently most unfavorable if the initial shape happens to be affine to this buckling shape. It is rather improbable that column as fabricated will be bowed exactly in such a symmetrical S-shape. Making such a simplifying assumption is, therefore, on the conservative side.

Assuming the initial and final shape of Fig. 6 (where, for simplicity, only the chords connecting the hinges have been shown instead of the curved column), one has

$$\frac{F}{3} L - P_e (d_o + d) = 0; \qquad F = k_{req} d$$

$$k_{req} = \frac{3 P_e}{L} (d_o/d + 1) \tag{7}$$

$$S_{req} = \frac{3 P_e}{L} (d_o + d) = k_{id} (d_o + d) \tag{8}$$

in view of Eq. (6).

Finally, for <u>three intermediate supports</u> it can be shown similarly that of the three possible modes again the zigzag shape shown schematically in Fig. 7 (consecutive supports displaced in opposite directions) governs, i.e. requires the largest rigidity of support. It is assumed that k is the same for all three supports, but this does not mean that the deflections d are the same. Taking moments about hinge "a" of the forces acting on spans 1 and 2, respectively,

$$P_e d_1 - (F_1 - F_2/2) L = 0$$

$$P_e (d_1 + d_2) - (F_2/2) L = 0$$

Introducing $F_1 = k_{id} d_1$ and $F_2 = k_{id} d_2$, the simultaneous solution of these two equations gives a quadratic equation whose larger root is

$$k_{id} = \frac{3.41 P_e}{L} \tag{9}$$

If this value is substituted in either of the two equations, one finds

$$d_2 = \sqrt{2}\, d_1 \tag{10}$$

which defines the ratio of the two deflections in the natural buckling mode.

If, for the same reasons as in the preceding case, one assumes the shape of the initially bowed column to be affine to this buckling mode, i.e. $d_{o2} = \sqrt{2}\, d_{o1}$, then the moment equations about the same hinge "a" in this column result in

$$k_{req} = \frac{3.41\, P_e}{L}\,(d_{o1}/d_1 + 1) = \frac{3.41\, P_e}{L}\,(d_{o2}/d_2 + 1) \tag{11}$$

$$S_{1\,req} = \frac{3.41\, P_e}{L}\,(d_{o1} + d_1) = k_{id}\,(d_{o1} + d_1) \tag{12}$$

$$S_{2\,req} = \frac{3.41\, P_e}{L}\,(\sqrt{2}\, d_{o1} + 2\, d_1) = k_{id}\,(d_{o2} + d_2) \tag{13}$$

Summary and Application

To summarize, it was found that for n equally spaced, equally rigid braces the values of k_{id} are as given in Table 2.

Table 2

Values of k_{id}

n =	1	2	3	4
k_{id} =	$\frac{2\, P_e}{L}$	$\frac{3\, P_e}{L}$	$\frac{3.41\, P_e}{L}$	$\frac{3.63\, P_e}{L}$

It is pointed out in the Appendix that these values agree identically with the rigorous theory of elastically supported columns, in fact, the value shown for n = 4, which has not been computed herein, has been taken directly from that theory.

Furthermore, in each of the analyzed cases it was found that for the initially bowed column

$$k_{req} = k_{id}\,(d_o/d + 1) \tag{14}$$

$$S_{req} = k_{req}\, d = k_{id}\,(d_o + d) \tag{15}$$

When the actual rigidity of bracing exceeds the required minimum value ($k_{act} > k_{req}$), as will usually be the case, then the deflection at a load almost equal to that which causes buckling between braces is found by substituting k_{act} for k_{req} in Eq. (14). Thus

$$d = d_o \frac{k_{id}}{k_{act} - k_{id}} \tag{16}$$

and in this case the required strength of the brace is

$$S_{req} = k_{act} d = d_o \frac{k_{id}}{1 - (k_{id}/k_{act})} \qquad (17)$$

(The last equation, incidentally, indicates that it is not sufficient to provide bracing of rigidity k_{id} as is sometimes maintained, because in that case the slightest initial crookedness d_o would require bracing of "infinitely large" strength in order that buckling do not occur below the stipulated value P_e).

These equations and values have been derived for prismatic columns with equally spaced and equal braces. This was done primarily for simplicity and also in order to demonstrate the relation of this approach to the exact theory of elastically supported columns (see Appendix). By the same simple means of writing moment equations about the fictitious hinges at the braces it is also possible to analyze the required bracing for columns with unequally spaced supports and/or with cross-sections which differ in the individual spans. In this case it is merely necessary to introduce for each portion between braces its appropriate P_e depending on length and cross-section of that portion.

In these cases of continuous columns the characteristics of the bracing so calculated are not exact but satisfy the stipulations (1) and (2) specified early in this paper to the effect that lower limits rather than precise values of these characteristics are desired. The method is equally applicable to the case of non-continuous columns, i.e. to members which at the brace locations are spliced or otherwise connected in a manner approaching a hinge. In fact, since hinges have been assumed at these locations, the analytical results are, if anything, more accurate for these discontinuous cases than for continuous columns.

The required bracing characteristics have been seen to depend, among other things, on the assumed initial bow d_o and on the additional deflection d deemed permissible under a load just below the buckling load, P_e. In design work P_e can be determined from Eq. (1) which, however, becomes difficult when L/r is less than about 100, since in that case the variable tangent modulus must be used. Instead, it is satisfactory in all cases to determine P_e as the allowable load multiplied by the safety factor for columns employed in the particular design code (about 1.9 to 2.0 in the A.I.S.C. Specification). The value of the imperfection d_o eventually might be specified in codes. Lacking such provisions one can be governed by permissible tolerances. Thus, in various parts of the Steel Construction Manual of the A.I.S.C. tolerances for crookedness and for out-of-plumb of columns and other compression members vary from about 1/500 to 1/1000 of the length. Assuming additional imperfections, such as eccentricities, it would seem consistent and realistic to specify d as about twice the value, i.e. 1/250 to 1/500 of the length depending on the particular type of member. As regards the additional deflection d, a rather rigorous requirement would be to make it equal to the assumed value of d_o. Since d is the deflection at incipient failure, this would mean that the deflection under design load would be less than half this value.

Practically, then, for a column of given characteristics and with given location of braces, one determines P_e and one specifies d_o and d from some such considerations as just illustrated. Then, from Table 2, one finds k_{id} and from Eqs. (14) and (15) the required rigidity of the bracing and the required strength if the actual rigidity happens to be equal to the required value. In most cases the actual rigidity will exceed this value, $k_{act} > k_{req}$; then correspondingly smaller actual deflections and required strengths are found from Eq. (17).

Continuously Braced Compression Members

In contrast to members braced at definite intervals there are cases where elastic (or near-elastic) support against buckling is provided continuously along the entire length of the member. A case in points was mentioned in the introduction, namely the compression chord of a roof truss which, without purlins, carries roofing spanning directly from truss to truss. A similar case concerns the bracing effect on the compression chord of open web joists obtained from the floor supported by the joist. As will be seen later, probably the most important application is the problem of lateral support for beams and girders.

It is not possible to treat this case by basic methods as elementary as those for point-support. However, the results of the well-established theory of columns on continuous elastic foundation (Ref. 5) can be approximated by convenient, algebraic expressions which will permit formulating requirements for such bracing in very simple terms.

An "ideal" elastic column on continuous, elastic foundation buckles into m half-waves, where the number m depends on the relative rigidities of the column and the foundation, and is obtained from the relation

$$m^2(m+1)^2 = \frac{\beta_{id} L^4}{\pi^4 E I} \tag{18}$$

where β is the modulus of the elastic foundation, and the subscript "id" indicates, as before, that the equation refers to the case of the ideal (straight, concentric, etc.) column. The load at which such buckling occurs is given by

$$P_{cr} = P_E m^2 + \frac{\beta_{id} L^2}{\pi^2 m^2} \tag{19}$$

where $P_E = \pi^2 EI/L^2$ is the Euler load of the same column without elastic support. For one extreme, namely for small values of β_{id}, the column bends into one half-wave and from Eq. (19), in non-dimensional terms

$$\frac{P_{cr}}{P_E} = 1 + \frac{\beta_{id} L^2}{\pi^2 P_E} \tag{19a}$$

At the other extreme, namely for very large values of β_{id}, the number m becomes so large that 1 (unity) can be neglected compared to it. Hence, an asymptotic relation for large m is obtained from Eq. (18), namely

$$m \rightarrow \frac{L}{\pi} \sqrt[4]{\frac{\beta_{id}}{E I}} \tag{18a}$$

Substitution of this expression in Eq. (19) gives, for very high foundation rigidity, the asymptotic value of the critical load, in non-dimensional terms

$$\frac{P_{cr}}{P_E} \rightarrow \frac{2}{\pi} \sqrt{\frac{\beta_{id} L^2}{P_E}} \tag{19b}$$

For values of m other than 1 or "very large as compared to 1" an exact value of the critical load can only tediously be obtained by computing m from Eq. (18) and substituting in Eq. (19).

However, the critical load for any value of β_{id} and, consequently, of any m can be obtained with entirely satisfactory approximation from the following simple relations:

$$\frac{P_{cr}}{P_E} = 1 + \frac{\beta_{id} L^2}{\pi^2 P_E} \quad \text{for} \quad 0 < \frac{\beta_{id} L^2}{P_E} \leqq 30 \tag{20}$$

$$\frac{P_{cr}}{P_E} = 0.6 + \frac{2}{\pi}\sqrt{\frac{\beta_{id} L^2}{P_E}} \quad \text{for} \quad \frac{\beta_{id} L^2}{P_E} \geqq 30 \tag{21}$$

It will be recognized that Eq. (20) is identical with the exact Eq. (19a) (for small values of β_{id}) and that for large values of β_{id} Eq. (19b) is the asymptotic value of Eq. (21), as it ought to be. Numerical computation shows that the error in determining P_{cr} from Eqs. (20) or (21) is in most cases extremely small and at the worst never exceeds ± 8% (see also Appendix).

By inversion one obtains the following relations which will be useful:

$$\frac{\beta_{id} L^2}{P_E} = \pi^2 \left(\frac{P_{cr}}{P_E} - 1\right) \quad \text{for} \quad 0 < \frac{\beta_{id} L^2}{P_E} \leqq 30 \tag{20a}$$

$$\frac{\beta_{id} L^2}{P_E} = \frac{\pi^2}{4}\left(\frac{P_{cr}}{P_E} - 0.6\right)^2 \quad \text{for} \quad \frac{\beta_{id} L^2}{P_E} \geqq 30 \tag{21a}$$

By analogy with Eqs. (14) and (16), which have been derived for point-supported columns, it will now be assumed without proof that for imperfect columns with continuous elastic support similar relations hold:

$$\beta_{req} = \beta_{id}\,(d_o/d + 1) \tag{22}$$

or by inversion

$$d = d_o \frac{\beta_{id}}{\beta_{act} - \beta_{id}} \tag{23}$$

These relations assume, as in the case of point support, that the initial shape of amplitude d_o is affine with the buckling shape. The relations are obviously exact in the limiting cases when $\beta_{act} = \infty$ (immovable support) or when $\beta_{act} = \beta_{id}$. For intermediate values the relations are conservative since complete affinity of initial and buckling shape is improbable.

With these relations one then obtains the strengths required per unit length of continuous bracing from the following equations (see Eqs. 15 and 17):

When $\beta_{act} = \beta_{id}$

$$s_{req} = \beta_{req}\, d = \beta_{id}\,(d_o + d) \tag{24}$$

and when $\beta_{act} > \beta_{id}$

$$s_{req} = \beta_{act}\, d = d_o \frac{\beta_{id}}{1 - (\beta_{id}/\beta_{act})} \tag{25}$$

Example. The four-panel top chord of a triangular roof truss is 20 ft. long and consists of two lipped, cold-formed channels, connected back to back, 3-1/2 x 4 in., 12 ga. According to A.I.S.I. Light Gage Cold-Formed Steel Design Manual, 1956, Table 4, p. 72, the relevant properties are: A = 1.69, $r_x = 1.38$, $r_y = 1.06$, $I_y = 1.90$ (all in inch units). The truss is perpendicularly braced only at mid-span and carries a roof deck which spans directly between adjacent trusses, without purlins. The deck is suitable connected to form a diaphragm which serves to brace the top chord against buckling perpendicular to the plane of the truss. What are the required characteristics of this continuous bracing? (For simplicity bending between panel points due to the uniform roof load will be neglected.)

In the plane of the truss, $L/r_x = 60/1.38 = 43.5$. For this value the allowable compression stress by A.I.S.I. Specification is 14.5 ksi. The safety factor for columns in that Specification is 2.17 so that the specified ultimate load of the top chord is

$$P_{ult} = 2.17 \times 14.5 \times 1.69 = 53.1 \text{ k.}$$

The Euler load of that chord for buckling out of the plane of the truss is

$$P_E = \pi^2 \times 29.5 \times 10^3 \times 1.90/(240)^2 = 9.6 \text{ k.}$$

For the bracing to be adequate, the buckling perpendicular to the truss should not occur at a load below that for buckling in the plane of the truss, i.e. $P_{cr} = P_{ult} = 53.1$ k.

Assuming an initial imperfection $d_o = L/500 = 240/500 = 0.48$ in. and a maximum permissible additional deflection of the same amount immediately before failure (both deflections being perpendicular to the truss), one has from Eq. (21a),

$$\frac{\beta_{id} L^2}{P_E} = \frac{\pi^2}{4} \left(\frac{53.1}{9.6} - 0.6 \right)^2 = 60.0 > 30 \quad \text{ok}$$

$$\beta_{id} = \frac{60.0 \times 9600}{(240)^2} = 10.0 \text{ lb./in./in.}$$

Then, from Eq. (22)

$$\beta_{req} = 10.0 \left(\frac{0.48}{0.48} + 1 \right) = 20.0 \text{ lb./ in/. in.}$$

For this value of β, from Eq. (24), the required strength of bracing

$$s_{req} = 20.0 \times 0.48 = 9.6 \text{ lb./ in.} = 115 \text{ lb./ft.}$$

Usually the actual rigidity will exceed the required value considerably. If, for example, $\beta_{act} = 100$ lb./in./in., then from Eq. (25)

$$s_{req} = 0.48 \frac{10.0}{1 - (10/100)} = 5.33 \text{ lb./in.} = 64 \text{ lb./ft.}$$

It is seen that very low values of both rigidity and strength of the diaphragm bracing suffice to stiffen the chord adequately. (An example of actual values for a light diaphragm is given later in this paper.) It should be noted that the very light bracing required in this case is due in part to the favorable relation of r_y to r_x. For less favorable ratios a stronger and stiffer bracing would be needed as can easily be checked by substituting for the top chord a pair of plain channels (without lips) of equal x-axis properties.

Lateral Bracing of Beams and Girders

The bracing of compression members is mostly achieved by bracing systems especially provided for the purpose, even though the preceding example illustrates a not infrequent exception from this situation. In contrast, for beams and girders the most frequent problem is that of deciding whether portions of the structure provided for other purposes give adequate, i.e. full lateral support. Thus, it is sometimes maintained that for a floor to furnish adequate bracing to the beam on which it rests, the compression flange of that beam must be embedded in a concrete slab. The moderate required values of strength and rigidity illustrated so far in this paper suggest that systems weaker than actual embedment in concrete may suffice for adequate lateral bracing of beams, such as floors or roofs made of a variety of prefabricated elements of lesser rigidity and strength than cast in situ concrete slabs. It is, therefore, desirable to develop a simple method for deciding whether a lateral system whose characteristics are known with reasonable accuracy provides sufficient bracing for a beam of given properties.

The analysis of the torsional-flexural buckling of "ideal" beams is considerably more involved than that of the simple, flexural buckling of columns. An analysis of such buckling for elastically supported beams has been developed only for a few special cases (Ref. 4) and is too complex for design use. However, for determining "safe lower limits (rather than exact values)" of the required bracing characteristics, as is the stated purpose of this investigation, conservative approximations can be introduced which lead to satisfactorily simple solutions.

It is known that a beam is more stable against lateral buckling than is its compression portion if regarded as an independent strut free to buckle in the direction perpendicular to the plane of the loads (Ref. 8). This can be understood qualitatively if one considers that the tension portion resists lateral bending and that, in addition, the torsional rigidity counteracts the twist which accompanies lateral beam buckling. These two influences, consequently, reduce the buckling tendency of the compression portion and increase its stability as compared to that of the identical but isolated compression member. At the same time numerical computations show that the total force in the compression portion of beams at the instant of lateral buckling, while it is larger than the Euler column load of that portion when isolated, is still of the same order of magnitude. Consequently, if bracing is dimensioned so that it is adequate for the compression portion assumed to be "cut off" from the tension portion, such bracing obviously will also be adequate, with some added safety, for the actual beam. This added safety, moreover, is hardly economically wasteful since the stability of the beam does not exceed that of the compression portion by a very sizeable amount and since, as will be seen again, relatively weak bracing suffices for full effectiveness.

The following discussion is restricted to I-shaped beams. Rectangular or double-web beams (such as U- or hat-shapes) are so stable that no lateral bracing is required for most any practical dimensions even though this is not explicitly recognized in some existing design specifications.

It is proposed, then, that bracing requirements for beams and girders be calculated in the following manner:

(1) Determine the total compression force of the fully braced beam at "incipient failure", i.e. when the allowable stress times the safety factor

is reached in the outer fiber of the beam so braced.

(2) By the methods previously developed determine the required characteristics for bracing against column buckling perpendicular to the plane of the loads of the compression portion alone, considered as isolated from the tension portion and as loaded by the above force.

No new analytical developments are needed for this approach which will be illustrated by an example, and further elaborated by another example in a later section of this paper.

Example. Determine the bracing characteristics for an 18 WF 50 beam of 30 ft. span, uniformly loaded by the floor which it carries, using the A.I.S.C. Specification (1949). For this beam $I_y = 37.2$ in.^{4}; d/bt = 4.22 in.$^{-1}$; flange area $A_f = 4.27$ in.^{2}; area of half-web $A_w = 3.02$ in.^{2} Without lateral bracing the allowable stress for this beam is 7.88 ksi. which is less than 40% of that for the fully braced condition (20 ksi.).

At "incipient failure", with the A.I.S.C. nominal safety factor of 1.65, the outer fiber stress of the braced beam is 1.65 x 20 = 33 ksi. (i.e. the yield point of A.S.T.M. A-7 steel). The total compression force is computed from the usual linear distribution of bending stresses. Neglecting the small difference between the stress at the outer and the inner fiber of the flange,

$$P_{ult} = 33 \times 4.27 + (33/2) \times 3.02 = 191 \text{ k.}$$

The Euler load for the compression half of the beam is

$$P_E = \pi^2 \times 29.5 \times 10^3 \times (37.2/2) / 360^2 = 41.8 \text{ k.}$$

Proceeding now exactly as in the case of the preceding example one finds, assuming $d_o = d = L/500 = 0.72$ in., $\beta_{id} = 12.5$ lb./in./in.; $\beta_{req} = 25$ lb./in./in. and for the case that $\beta_{act} = \beta_{req}$ then $s_{req} = 18$ lb./in. = 216 lb./ft. As will be seen later, the rigidity even of quite flexible floor systems is likely to exceed β_{req} considerably. If one assumes, for example, $\beta_{act} = 200$ lb./in./in. one obtains $s_{req} = 9.6$ lb./in. = 115 lb./ft. These, then, are the characteristics which assure bracing adequate for the unreduced bending stress of 20 ksi. to be permissible, instead of 7.88 ksi. for the case of no, or inadequate, bracing.

A uniformly distributed load of 1280 lb./ft. is required to develop this permissible stress or, to put it differently, a load of 1.65 x 1280 = 2120 lb./ft. will produce that state of "incipient failure" or extreme fiber yielding for which the bracing has been proportioned. It is seen that the required lateral support, s_{req}, in this example does not exceed 10% of the vertical load. Hence, mere friction between the top of the beam and the floor which rests on it would suffice to develop the force required for lateral bracing. This is not to suggest that such friction should actually be relied upon; however, the weakest connection which is practically feasible, say 1/2 in. bolts at 3 ft. spacing along the span, would develop a multiple of the required holding force.

Sample Characteristics of a Floor System

Not long ago a sizeable number of tests were made at Cornell University by A. H. Nilson on full-size cellular steel panel floors to ascertain their effectiveness as diaphragms, primarily for wind and earthquake bracing. The characteristics measured in these tests are, of course, the very same ones

which matter if such floors are to act as bracing for the supporting beams. To give some indication of actual values of horizontal strength and rigidity for floors of this kind it will suffice to quote and interpret the results of one of the weakest systems tested in that series.

Omitting irrelevant details, the test set-up is schematically shown in plan on Fig. 8a. Three panels of steel framework, 10 x 12 ft. each, were erected in the laboratory. Eight short stub columns, all placed on rollers, carried the shown system of beams (12 WF 27 longitudinally and 10 WF 21 transversely). The two outer transverse beams were anchored at their ends A and D; hydraulic jacks with load cells were placed at B and C to apply horizontal forces to the inner transverse beams. Horizontal deflections were measured at E and F. Since the longitudinal beams were not continuous across the columns, the framework itself was so flexible horizontally that by pushing at B or C by hand a deflection of an inch and more could easily be produced. Cellular steel floor panels of one of the widely used makes were then erected, simulating standard practice. The individual panels, 2 ft. wide, are shown in dashed lines; they were 3 in. deep, of 16 ga. steel, and were welded to the beams. In the particular test being discussed, adjacent panels, along their hook-joints, were connected to each other by crimping at 2 ft. intervals by means of a clinching tool. Tests have shown that diaphragms of this type, together with the surrounding beams, act somewhat like plate girders, and develop considerable horizontal strength and rigidity.

Loads of jacks B and C were kept equal throughout the test, and horizontal deflections at E and F proved equal within a few thousandths of an inch. The load-deflection diagram is given on Fig. 8b. It is seen that it is practically linear up to a load, per jack, of 12,500 lb. at which load the deflection was 0.125 in. Consequently, over the linear range the spring constant was k_{act} = 100,000 lb./in. The diaphragm failed under a horizontal load per jack, S_{act} = 24, 400 lb.; even if, conservatively, one were to define the useful strength as the limit of the elastic range, one would have S^{el}_{act} = 12,500 lb.

When, in another test, the usual 2-1/2 in. concrete fill was applied over the steel deck, the rigidity increased to k_{act} = 1,150,000 lb./in., the ultimate strength to S_{act} = 57,000 lb., and the limit of the elastic range to S^{el}_{act} = 24,000 lb.

To see what these values mean, a floor system will be assumed identical with that tested, except that a 30 ft. girder from A to D will be assumed to carry the transverse beams BE and CF without intermediate support (i.e. on a 30 ft. span) and that similar transverse beams frame into that girder from the other side. It is easily checked that the 18 WF 50 section of the preceding example is then satisfactory to carry, for such a system, a total floor load of 100 lb./sq. ft.

The girder, then, is stiffened at the B and C by the transverse beams who, in turn, are supported against motion in direction of their axes by the strength and rigidity of the diaphragm formed by the panels which they support. The unsupported length of the girder is now reduced to 10 ft., resulting in Ld/bt = 507 600, so that by present A.I.S.C. Specifications the unreduced stress of 20,000 psi. is permissible. Making use of the data of the preceding example, the girder is then to be braced for a force $P_{ult} = P_e$ = 191 k. From Table 2, for n = 2,

$$k_{id} = 3 \times 191{,}000 / 120 = 4770 \text{ lb./in.}$$

Assuming again $d_o = d = L/500 = 360/500 = 0.72$ in., one has from Eq. (14)

$$k_{req} = 4770\ (0.72/0.72 + 1) = 9540 \text{ lb./ in.}$$

and if, by chance, $k_{act} = k_{req}$, then from Eq. (15)

$$S_{req} = 9540 \times 0.72 = 6860 \text{ lb.}$$

However, the actually measured rigidity was $k_{act} = 100{,}000$ lb./in. so that, from Eq. (17), for this rigidity

$$S_{req} = 0.72\ \frac{4770}{1 - (4770/100{,}000)} = 3620 \text{ lb.}$$

A comparison of these required values with the corresponding quantities obtained in the test with bare deck shows that the actual rigidity k was about 10 times the required amount, and the actual usable strength S_{act}^{el} about 3.5 times the required amount. It is seen, then, that this laterally relatively weak and flexible floor system supplies a multiple of the support required in this example for full lateral bracing; an even greater reserve obtains after the usual concrete fill has been applied.

(It should be observed that all the foregoing determinations have been made for the state of incipient failure, rather than for design loads. This is necessary because some of the involved phenomena are non-linear. It follows from this that the required bracing characteristics, so calculated, already involve a safety factor which is, in fact, greater than that implied in the pertinent general design specification. This is so because the proposed analysis, as repeatedly emphasized, is based on a number of conservative assumptions. Hence the calculated, required properties (k_{req}, S_{req}, etc.) refer to the actual minimal requisite values and are not in need of further multiplication by a safety factor.)

Reservations and Limitations

(1) It has been suggested that the equivalent initial crookedness, duly related to applicable straightness tolerances, might be assumed of the order of L/250 to L/500 and that the additional deflection under a load just below failure might, conservatively, be taken as equal to this amount. The three numerical examples have been computed on that basis. While the writer believes these quantities to be reasonable, it is evident that they should be related to imperfections and, implicitly, to safety factors assumed in the relevant design specifications. The quoted, specific values should, therefore, be understood rather as samples subject to more detailed determination by those reponsible for the design specification.

(2) The proposed method for bracing of beams and girders utilizes the proposition that a beam is more stable than its compression portion if the latter were isolated from the rest of the member. While this is true in the vast majority of cases, contrary situations may exist, for instance when the center of gravity of the load is located at a level very considerably above the top flange of the beam (see Refs. 8 and 9). (As an illustration, consider the unlikely case of an entirely detached brick wall supported on the top flange of an unbraced girder.) However, in these cases, and for the same reasons, the usual, simplified design code provisions for lateral buckling are also inapplicable.

(3) For the time being the proposed method should be applied only in

connection with elastic design, since it has been developed to provide adequate support up to loads which cause either incipient yielding or buckling in a plane perpendicular to the bracing. It is known, in regard to plastic design of steel structures, that the ultimate loads predicted by that method will not be reached unless the structures are adequately braced at least at those locations where yield hinges form (Ref. 10). This condition is not included in the foregoing development. Hence, when applied to plastic design the proposed method of bracing analysis may require modifications which are not a subject of this paper.

CONCLUSIONS

Experimental evidence is cited to show that very light lateral bracing (small strength and small rigidity) suffices to produce very large gains in carrying capacities of columns and beams.

A simplified analysis of initially imperfect columns with one or more intermediate supports leads to a method which permits one by the simplest of calculations to determine lower limits for the strength and rigidity required to produce "full bracing". In regard to continuous lateral support, by using two simple equations to express the results of the rigorous theory of columns on elastic foundations, a similarly simple method is developed for determining the bracing characteristics for this case.

By an approximate method which errs on the conservative side, the problem of laterally supported beams is reduced to that of laterally supported columns. By this means a simple procedure is developed for determining minimum characteristics for lateral support of beams and girders.

Sample computations are given; comparison with bracing characteristics determined in full-size tests on a light floor system show that, in particular, for beams and girders lateral support much lighter than that provided by flange embedment in a concrete floor is generally adequate to furnish adequate bracing.

An appendix shows the relation of these approximate methods to the results of the exact theory of columns on elastic supports or on elastic foundation.

APPENDIX

Throughout this paper repeated reference has been made to the results of the rigorous analysis of ideal columns on elastic supports. To elaborate briefly on those features, these results are given in a graphical non-dimensional presentation, Fig. 9, which has been taken from Refs. 3 and 6. The value P_E used in the non-dimensional abscissae is the Euler load of the identical, full length column without intermediate supports. Consequently, this definition is identical in the elastic range with P_E as used before for continuous support in Eqs. (19) to (21). For point support, the value P_e, Eq. (1), had been referred to the free length between supports; hence, for n intermediate equidistant point supports the relation between P_e and P_E is

$$P_e = P_E/(n + 1)^2 \qquad (26)$$

For a column elastically supported at n equidistant points it is seen that the critical buckling load increases with increasing support rigidity k. For

very low values of k the column buckles in a single half-wave (m = 1); the larger k, the larger the number m of half-waves and the larger P_{cr}, up to that value of k at which the number of half-waves becomes equal to the number of free lengths between point-supports, (i.e. $m_{max} = n + 1$). For any magnitude of k equal to or larger than this particular value the column buckles as if it were held immovably at the intermediate supports, in the manner shown on Fig. 1 for n = 1. For such elastic support, then, the ideal column has attained its maximum possible carrying capacity, which represents "full bracing" by the definition of this paper. These particular minimum values of k which produce "full bracing" by the rigorous theory are identical with those values k_{id} of Eqs. (3), (6), (9) (also Table 2) which have been determined by elementary means in this study. They are shown on Fig. 9.

For continuous, elastic foundation the solid curve of Fig. 9 gives the results of the rigorous theory and indicates, again, that as the rigidity of the foundation increases, so do the number of half-waves m as well as the theoretical buckling load P_{cr}. Of the two expressions Eqs. (20) and (21) used in this study to represent the relationship between P_{cr} and β_{id} the first, up to $\beta_{id}L^2/P_E = 30$ is identical with the exact relation over this range. The second, approximate expression, Eq. (21), is shown by the broken curve. Agreement entirely satisfactory for engineering purposes is seen to obtain throughout, with maximum deviations of about ± 8% confined to a few short regions.

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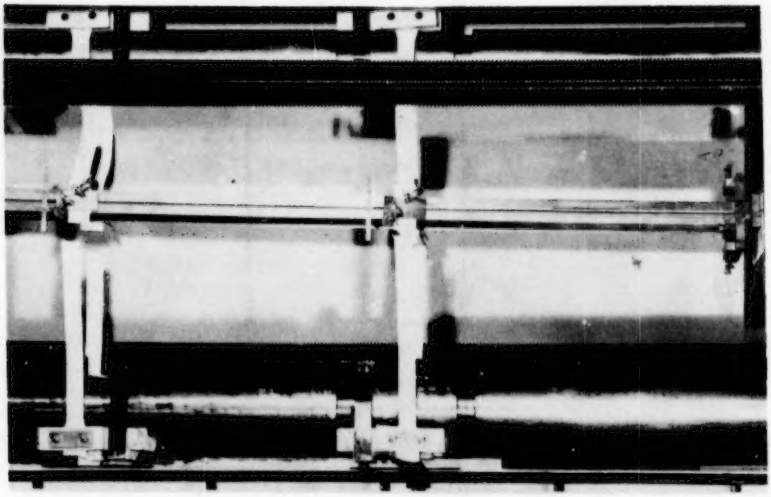


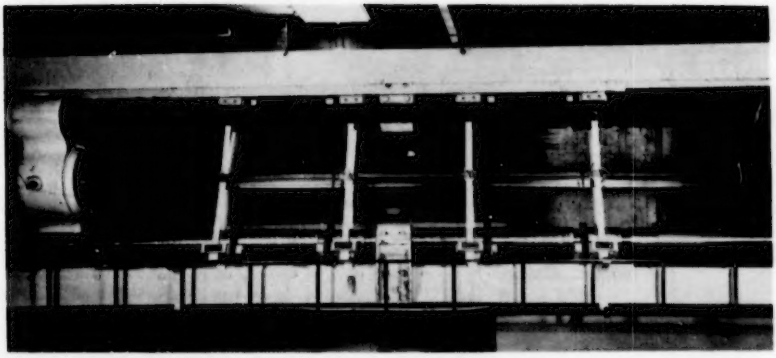


FIG. 1

TEST OF STEEL STUD BRACED BY CARDBOARD STRIPS

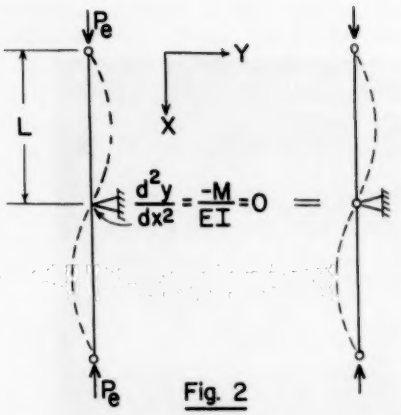


Fig. 2

Fig. 3

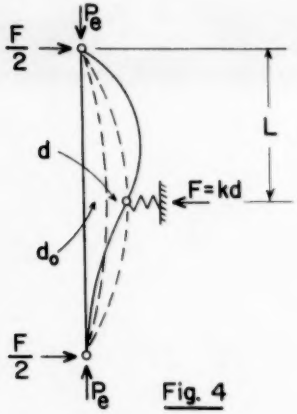


Fig. 4

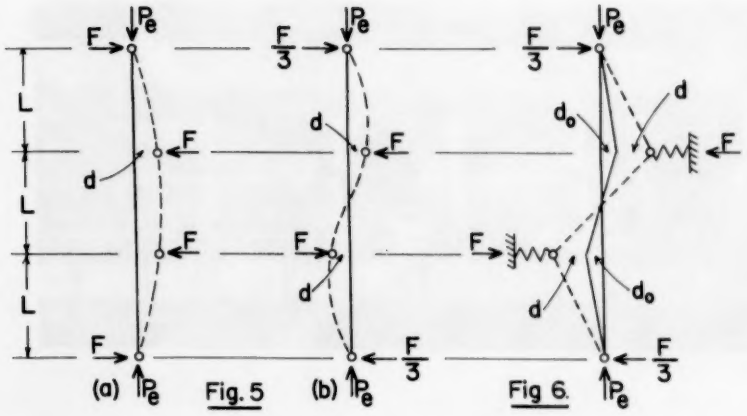


Fig. 5

Fig 6.

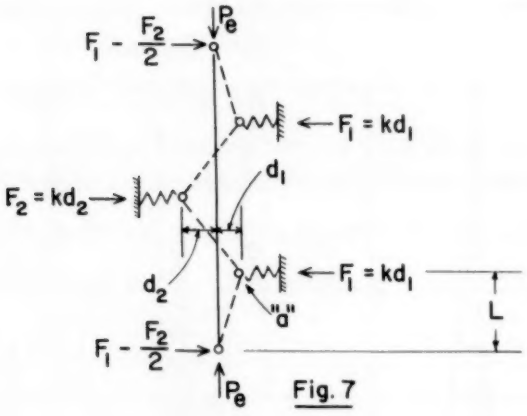
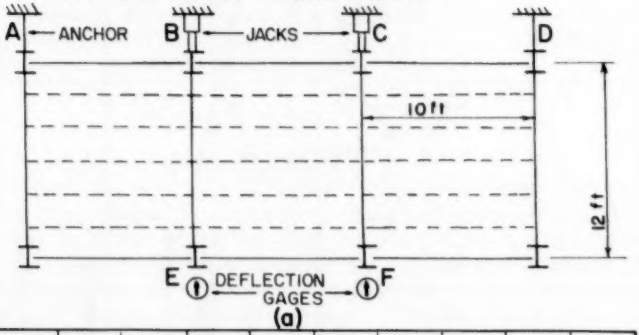
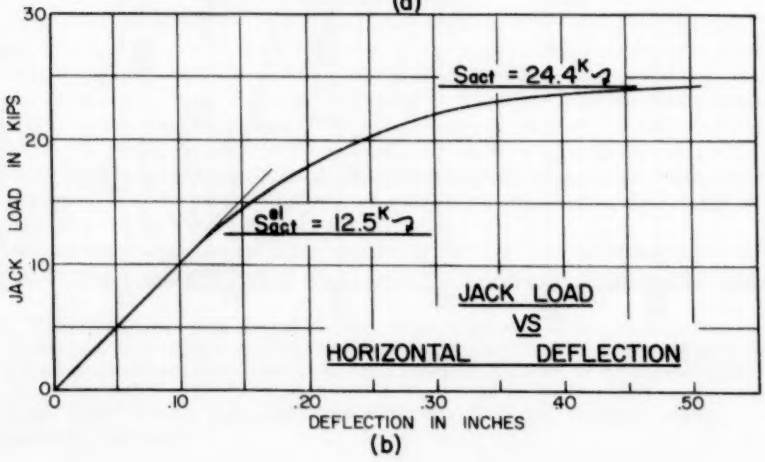


Fig. 7

Fig. 8 FULL SCALE TEST ON CELLULAR STEEL FLOOR



(a)



(b)

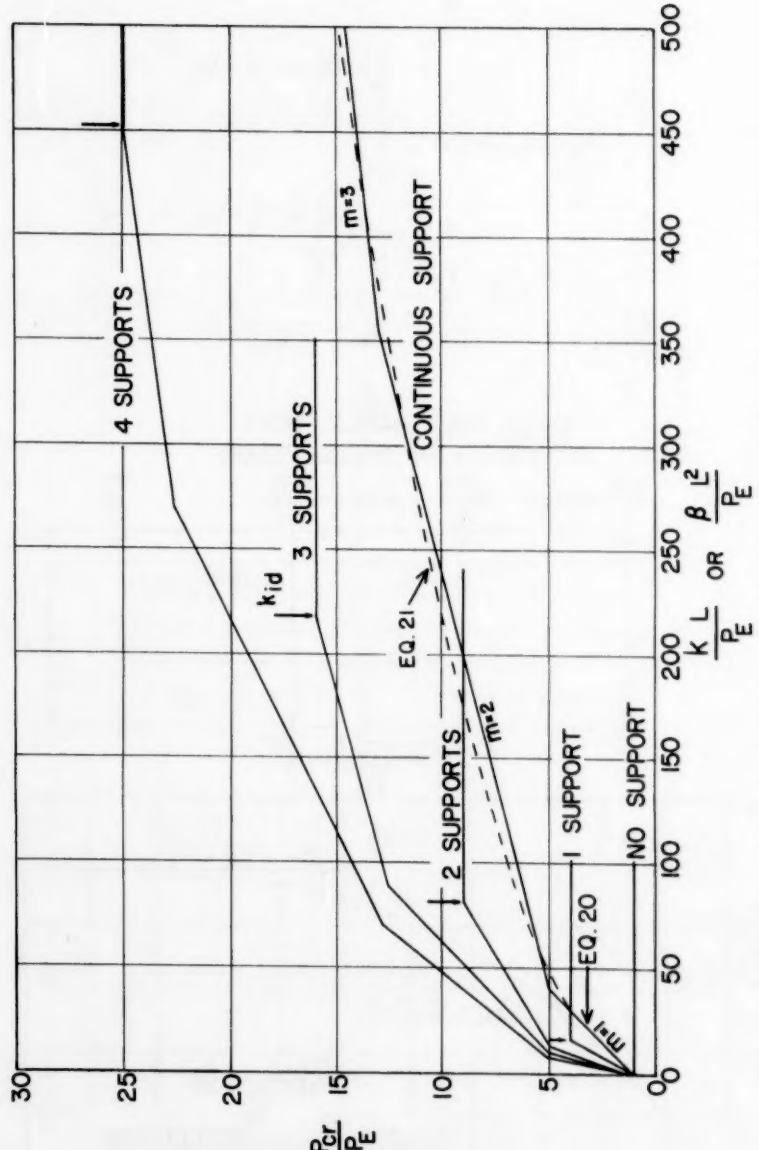


Fig. 9 CRITICAL LOADS FOR ELASTICALLY SUPPORTED COLUMNS

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NUMERICAL SOLUTIONS FOR BEAMS ON ELASTIC FOUNDATIONS

Henry Malter,[1] A.M. ASCE
(Proc. Paper 1562)

SYNOPSIS

Two numerical methods for the solution of beams on elastic foundations are considered. The first, based on the Newmark Method,(1) is a step by step integration process. The second method, using finite difference equations obtains a solution by means of a set of simultaneous equations. Examples are solved, and comparisons made between the results obtained using various approximate as well as rigorous methods.

INTRODUCTION

Beams on elastic foundations confront the analyst with three phases to each problem. The first, and by far the most important concerns itself with the basic assumptions of the behavior of the structure—the type of foundation. The second phase is that of the selection of the constants needed—modulus of elasticity, moment of inertia, foundation modulus. The last phase of the problem consists of the mathematical manipulation of the information obtained from the first and second phases. This article will concern itself only with the latter problem—the mathematical manipulation. The mathematics may be either "rigorous," or "approximate." The approximate methods may be either "graphical" or "numerical." A numerical method was employed by Levinton, M. ASCE,(2) whereas Popov, M. ASCE,(3) used a graphical method to obtain approximate solutions. In this article two "numerical" methods will be employed, and the results compared with those obtained by other approximate as well as by rigorous methods.

Two assumptions may be made regarding the type of foundation encountered. The more general one is that of an elastic solid involving complete continuity.

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This type of problem, however, does not lend itself easily to mathematical solution, and will not be considered in this article.

A second type of elastic foundation, and one that does lend itself easily to approximate solution is one that is known as a "Winkler" foundation. This latter type is characterized by the fact that the pressure in the foundation is, at every point, proportional to the deflection occurring at that point, and is unaffected by deflections at adjacent points. This idealized foundation may be thought of as consisting of a series of independent springs. This subject has been excellently discussed by Hetenyi in his book "Beams on Elastic Foundations,"(4) the most thorough coverage of the problem in the English language. Prof. Popov in his aforementioned article, has an excellent summary of the problem, and in addition has a lengthy bibliography of the subject in both the Russian and the German languages.

A "rigorous" method should at all times be sought after. Hetenyi has collected a wealth of information to simplify the solution by rigorous methods for various conditions of foundation and of loading. But despite the simplifications, and despite the use of derived formulas, as well as the use of prepared tables, it becomes evident very soon that a great deal of time and labor is required for all but the simplest type of problem.

The designer has thus been confronted with the various stages of the problem. First, should it be assumed that a Winkler type foundation is applicable. This leads to the first approximation. Secondly, what are the values of the various constants employed—modulus of elasticity and moment of inertia of the supporting beam, and spring constant or foundation modulus of the supporting material. This leads to a second set of approximations. And now finally, after having accepted and recognized the above approximations, the designer is confronted with the choice of which mathematical manipulation to employ—"rigorous" or "approximate." Recognizing the nature of the entire problem, it is often preferable to use an approximate method, and to quickly obtain a solution—one that may in actuality be no further removed from a "ture and exact" solution than that derived by a "rigorous" method.

Two numerical methods are discussed, both of which are based on the well known relationship between moment area and angle change. In the first, using the Newmark Method, a step by step method of numerical integration is employed. In the second, using the finite difference method, a series of simultaneous equations is solved. Both methods are quick and general, although the finite difference method seems to be less time consuming.

One should not be tempted by ease of mathematical manipulation to apply the above mentioned methods without careful consideration of the limits of any problem encountered. Where the supporting media is water, or elastic beams, or even piles driven to a solid supporting material, one may find direct application of a Winkler type of foundation. Where soil is the supporting media, there is no substitute for a good knowledge of soil mechanics, and any method of solution must be employed with caution.

Newmark Method

A general knowledge of the Newmark Method is assumed, and the details will be reviewed but briefly. Foundation problems may consist of concentrated loads, single point spring reactions, distributed loadings, and distributed foundation reactions. In a step by step procedure the distributed loadings and

distributed foundation reactions are converted into concentrations at the subdivisions selected. The use of concentrations introduces an error inherent in the system. Increasing the number of divisions reduces the error, but increases the amount of work. In the problems to be considered the subdivisions will be made equal to each other. This is not necessary, but simplifies further the arithmetic of the process through the use of the common multiplier involving the length of the subdivision "h," (usually shown in the extreme right column).

$$R_A = \frac{h}{24}(7A + 6B - C)$$

$$R_B = \frac{h}{12}(A + 10B + C)$$

$$R_C = \frac{h}{24}(7C + 6B - A)$$

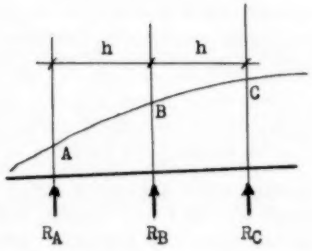


Fig. 1 - Distributed Loads Converted To Concentrations

A suggested method for changing distributed loads to concentrations is indicated in the accompanying sketch (Fig. 1). The formulas utilized by Prof. Newmark assume that the loading curves are parabolic and continuous, and for the examples discussed are applicable. For other types of problems straight line formulas may be employed, or account taken of discontinuities (see example 2).

The designation of the terms and the sign convention will be as follows: - all terms positive as indicated; (a) load "p," upward; (b) shear "V," upward force on left of section; (c) moment "M," compression in upper fiber; (d) angle change "α," causing increase in slope; (e) slope "ϕ," downward to right; (f) deflection "y," downward; (g) coordinates—add to right. Distributed values will be designated by the terms "p," "q," and "α," whereas the concentrations will be shown as "$\bar{p}$," "$\bar{q}$" and "$\bar{\alpha}$"

The numerical procedure to be outlined is a method of arithmetical integration based on the well known relationships:

a) $dV = p\,dx$

b) $dM = V\,dx$

c) $\alpha = \frac{M}{EI}$

d) $d\phi = \alpha\,dx$

e) $dy = \phi\,dx$

Correction Configurations

Before starting with the given loadings, a set of "correction configurations" must be established. These will be required later in the general solution. The correction configurations will be derived from two sets of corrections introduced into the problem—the first involving a uniform deflection, and the

second a uniform rotation. To obtain these configurations one should proceed as follows:

a) Assume a uniform deflection at each panel point of the beam "y_a."

b) Using the assumed deflections determine the resisting pressures at each panel point "q."

c) Change distributed reactions to concentrations.

d) Assuming zero reaction at the left support, determine the shear "V" along the beam by adding successively the various concentrations. There will usually be an unbalanced shear at the extreme right.

e) Assuming zero moment at the left support, determine the moments "M" along the beam by adding successively the various shears. There will usually be an unbalanced moment at the extreme right of the beam.

Beginning anew assume a uniform rotation of the beam. Any point may be considered a center of rotation, but for convenience assume the center of the beam as the center of rotation. The uniform rotation will result in a set of deflections. Using these deflections repeat the above process, again obtaining an unbalanced shear and unbalanced moment at the extreme right of the beam.

Two correction configurations will be required for the general problem. The first "A" should have zero unbalanced end shear, but will have some unbalanced moment. The second "B" should have zero unbalanced end moment, but will have some unbalanced shear. Both may be obtained by proper combinations of the "uniform deflection" and the "uniform rotation" corrections described above.

General Solution

After obtaining the correction configurations, the steps involved in obtaining a general solution are as follows:

a) Assume a deflection "y_a" at each division point of the beam. This assumed deflection should be estimated using whatever information is available.

b) Using the assumed deflections, determine the resisting pressures along the beam.

c) Change all values of "q" to concentrations "$\bar{q}$," and combine with given loadings to obtain total load on the beam.

d) Assuming zero reaction at the left support, determine the shear "V" along the beam. There will normally be an unbalanced shear at the right end of the beam.

e) Assuming zero moment at the left support determine the moments along the beam. There will normally be an unbalanced moment at the right end.

f) For static equilibrium the unbalanced shear and the unbalanced moment should both be zero (no end support, no end restraint). To account for the first, add enough of configuration "B" to make the unbalanced shear zero. This is line "M_{c1}."

g) To satisfy the condition of no end restraint, add enough of configuration "A" to make the end moment zero. This correction line is "M_{c2}."

h) Add $M + M_{c1} + M_{c2} = M_a$. The values of "$M_a$" thus obtained correspond to some certain deflection of the beam that produces static equilibrium. The moment values obtained are the panel point values of a distributed moment along the beam.

i) The distributed moment along the beam results in a concentrated angle change at each panel point "$\bar{\alpha}$."

j) Assuming any slope at the left end, proceed to find the slope "ϕ" at all other points.

k) Assuming any deflection at the left end, proceed to find the deflections along the beam by adding successively the slopes encountered.

l) Change "y" to "y' " by multiplying all values of "y" by the common multiplier in the right hand column. There is thus obtained a series of deflections that is directly in inches (or some convenient unit).

m) Using the values of "y' " obtained, proceed as in steps "a" to "k" to obtain a value for "M_o."

This completes one cycle of computations. The cycle consisted of assuming a set of deflections, computing shears and moments, correcting the moments, (and therefore the assumed deflections) to bring the beam into static consistency. The statically consistent moment "M_a" is then applied to the beam, and the deflection of the beam determined, again bringing it into static consistency. The resulting deflections correspond to a set of moments. These final moments "M_o" should be identical with those of the originally corrected moments "M_a." If they are not, (and they will rarely be even in close agreement after one cycle) a second cycle of computations must be begun. It is often found that "M_o" is a better assumption than "M_a" was, and should now be used as a new starting point. The agreement after the second cycle should be much closer than after the first. After several cycles sufficient agreement will be found between "M_a" and "M_o" to warrant an acceptance of the results. When sufficient accuracy has been obtained, the "q" line of the last cycle should be corrected using the same percentages employed in the correction of the final moments, and thus a "q" corrected line established.

The rapidity of convergence is a function of the relationship of the relative stiffness of the beam (E I) and that of the foundation modulus (spring value). Where the beam is much "stiffer" than the foundation, convergence is rapid, and "M_o" is always a much better approximation than "M_a" was. When the relationship is reversed, divergence will occur, and the problem cannot be solved by ordinary means. For the intermediate case of the spring constant being equal to or close to the "critical spring constant," convergence is slow, and the average of "M_a" and "M_o" should be used as the new value of assumed moment.

The above outlined procedure should be recognized as a numerical systematization of the Vianello-Stodola[5] method, and follows essentially by numerical methods the graphical outline pursued by Prof. Popov.

Finite Difference Method

In all elementary textbooks on Mechanics of Materials there appears the following set of equations for the solution of beam deflection problems.

a) $y =$ deflection

b) $\frac{dy}{dx} = \phi$ slope

c) $\frac{d^2y}{dx^2} = \frac{M}{EI}$ moment

d) $\frac{d^3y}{dx^3} = \frac{V}{EI}$ shear

e) $\frac{d^4y}{dx^4} = \frac{w}{EI}$ load

In the above, one should remember that both E and I must be constant quantities for general use. With the moment of inertia I variable, as it often is, the "moment" term is applicable, but the "shear" and "load" equations become more complicated.

As in many problems involving differential equations, it has been found that difference operators may be substituted for the differential operators. Using the "molecule notation for central difference operators"(6) one obtains the following:

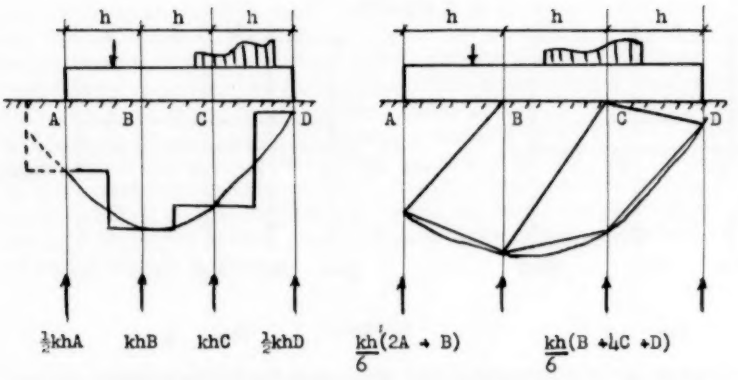
$$\frac{\Delta}{\Delta x} = \frac{1}{2h}(-1, 0, +1) \qquad \frac{\Delta^3}{\Delta x^3} = \frac{1}{2h^3}(-1, 2, 0, -2, 1)$$

$$\frac{\Delta^2}{\Delta x^2} = \frac{1}{h^2}(1, -2, 1) \qquad \frac{\Delta^4}{\Delta x^4} = \frac{1}{h^4}(1, -4, 6, -4, 1)$$

These difference operators may be used in various ways for the solution of elastic foundation problems.

A beam supported along its entire length by an elastic medium and subjected to vertical forces, will be deflected in some manner, and will result in a foundation pressure that will vary along the length of the beam. This foundation pressure that is most likely a smooth continuous function, may be represented in various ways. First, one may assume a "stepped" variation (Fig. 2a) resulting in the subgrade reactions 1/2 khA, khB, khC, etc., where "k" is the foundation modulus, "h" is the length of the subdivision, and the letters "A," "B," "C," etc. are the deflections of the beam at the various subdivisions. Using a "straight line" variation (Fig. 2b) would result in the more accurate form of concentrations $\frac{kh}{6}(2A + B)$, $\frac{kh}{6}(A + 4B + C)$, etc., but would result in more labor. The parabolic formula suggested by Prof. Newmark for his numerical integration process could be employed to give concentrations of the form $\frac{kh}{12}(A + 10B + C)$. The simplest method to employ is the "stepped" method, and it is the one that will be used in the solution of the examples below. (Fig. 2)

The most convenient formula to use in the solution of elastic beam problems is that for moment $\frac{d^2y}{dx^2} = \frac{M}{EI}$. In difference notation this becomes $\frac{1}{h^2}(A - 2B + C) = \frac{M_B}{EI}$, $\frac{1}{h^2}(B - 2C + D) = \frac{M_C}{EI}$, etc. The moment at the point under consideration may be found by including all of the given loads as well as the unknown foundation concentrations either to the left of the point, or to the right. Thus, considering the simple problem of a beam supporting a single



(a) Stepped (b) Straight Line

Fig. 2 - Foundation Pressure Concentrations

concentrated load and resting on an elastic foundation (Fig. 3), it can be seen that the values of the moments at point B will be

$$M_B = -\tfrac{1}{2}kh^2A + Ph \qquad \text{(left)}$$

$$M_B = -\tfrac{1}{2}kh^2C \qquad \text{(right)}$$

where, as before, the foundation modulus is "k," the division spacing is "h," the given load "P" and the assumed deflections "A," "B," and "C" (all in the same units). The three simultaneous equations needed to solve this problem would be obtained by using the following: (1) moment at B from the left; (2) moment at B from the right; (3) $\Sigma V = 0$. Any consistent sign convention may be employed. The three resulting equations would be

$$1)\quad A - 2B + C = -\frac{kh^4}{EI}\left(\tfrac{1}{2}A\right) + \frac{Ph^3}{EI}$$

$$2)\quad A - 2B + C = -\frac{kh^4}{EI}\left(\tfrac{1}{2}C\right)$$

$$3)\quad \tfrac{1}{2}A + B + \tfrac{1}{2}C = \frac{P}{kh}$$

A solution of the above equations should give the deflections directly, and from the deflections one could obtain the foundation pressures. It should be noted that the first two equations imply zero moment and zero shear at both ends of the beam. Combined with the third equation, one can see that the laws of statics must be satisfied. The above method is similar to that employed by Joseph Gold, Jun. M. ASCE.(7)

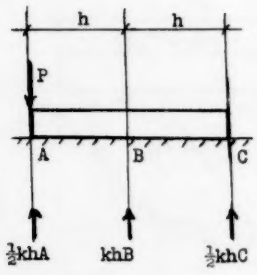


Fig. 3

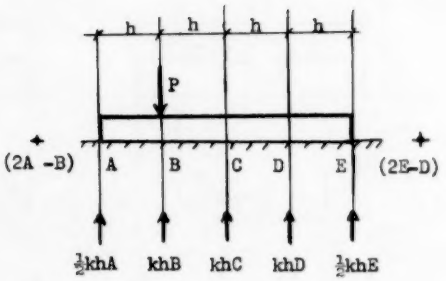


Fig. 4

A method, not recommended, but of interest for its theoretical implications would be one involving the use of the fourth difference equation. This equation having five terms, requires the same number of subdivision points (Fig. 4). Fictitious points at each end may be used, and can be derived from the fact that the moment at the end of the beam is zero, thus obtaining the relationship

$$B' - 2A + B = 0$$

$$B' = 2A - B$$

This method, however, does not lend itself easily to the solution of problems involving variable moment of inertia, and will in general be more laborious than that using the second order equations. To indicate the method, the five simultaneous equations that would be required to solve the problem of figure 5 would be

1) $(2A - B) - 4A + 6B - 4C + D = -\frac{kh^4}{EI}(B) + \frac{h^4}{EI}\left(\frac{P}{h}\right)$

2) $A - 4B + 6C - 4D + E = -\frac{kh^4}{EI}(C)$

3) $B - 4C + 6D - 4E + (2E - D) = -\frac{kh^4}{EI}(D)$

4) $\frac{1}{2}A + B + C + D + \frac{1}{2}E = \frac{P}{kh}$

5) $A - 2B + C = -\frac{kh^4}{EI}\left(\frac{1}{2}A\right)$

Recognizing that the $\frac{P}{h}$ term is the distributed equivalent of the concentrated load P, the first three equations are straightforward. The fourth implies that $\Sigma V = 0$. Inasmuch as the two fictitious terms used imply no external moment, one additional equation is required to eliminate the possibility of end shear. The fifth equation assumes zero end shear at the left end, and in consequence, zero shear at both ends. Thus, the laws of statics are satisfied.

This method may at times be useful, although it is not recommended for general use.

The application of both numerical methods to the solution of various examples can be best illustrated by the solution of the same set of problems that were used by Prof. Popov in his article. These problems have the value of being of various types. The first employs a variable foundation modulus. The second is symmetrical, but with varying moment of inertia. The third involves unsymmetrical loading. A fourth example has been added to demonstrate the handling of concentrated moments. All four examples will be solved by the finite difference method, whereas only the first and second will be solved by the Newmark Method.

Example 1

The first example is one that is taken from Hetenyi, "Beams on Elastic Foundations" (page 111), and consists of a weightless beam 8" by 6" with equal loads of 100 lbs. at each end. The constants for the beam are $E = 2.5 \times 10^6$ psi and I = 144 inches.(3) The foundation modulus is assumed to vary linearly from 87.5 lbs. per cubic inch at the left end to 12.5 lbs. per cubic inch at the right end, or from 700 lbs. per lineal inch (pli) at the left end, to 100 pli at the right end. The dimensions of the beam are shown in Table 1A for the Newmark Method, and also in Table 4 for the finite difference method.

To estimate the order of magnitude for the initial assumption of beam deflection to be used in the Newmark Method, it can be seen that with an average uniform modulus of 50 lbs. per cubic inch, and a rigid beam, that the average deflection would be 0.004 inches. Obviously the deflection should be greater under the loads, and greater at the right end than at the left. Using three subdivisions of 40 inches each, the initial assumed deflections are as shown from Table 1A

y_a	+6	+1	+3	+10	10^{-3} inches

The end of the first cycle shows the following comparisons:

M_a	0	-17.0	-19.3	0	$\frac{h^2}{12}$
M_o	0	-14.8	-17.7	0	"

Obviously the agreement is not sufficient. A second cycle was run giving the following values:

M_a	0	-15.0	-18.0	0	$\frac{h^2}{12}$
M_o	0	-15.2	-17.9	0	"

Sufficient accuracy having been obtained, the foundation pressures were computed for the last cycle, and are shown in Table 1B. In Table 2 there is shown the computations for the correction configurations required for this example.

The solution by the finite difference method is shown in Table 4. A stepped varying foundation modulus is used with 650 pli at the left end, and 150 pli at the right end. The results of the various methods are also shown in Table 4, and the close agreement can be readily seen.

Example 2

This example is one taken from those solved by Prof. Popov, and consists of a continuous concrete footing having a cross section as shown in Table 3A, and loaded with a line load of 150 kips per ft. The weight of the footing is neglected. The pertinent values are E = 432,000 kips per sq. ft., and the foundation modulus is 300 kips per cu. ft. The moment of inertia is a function of the beam depth and varies along the length of the beam.

In this problem advantage is taken of the symmetry, and only one correction configuration employed for the Newmark solution—that for shear. The discontinuity at point C in the value of the moment of inertia involved a process of obtaining concentrated angle changes on each side of the point, and then adding both of them to obtain the total angle change at the point. After three cycles a comparison of the M_a and the M_o values indicated sufficient accuracy and the values of the foundation pressures were computed, and are shown in Table 3B.

The finite difference solution is shown in Table 5. For this method an average value of moment of inertia was used at the point of discontinuity, and is shown in the diagram of Table 5. There is also included in this Table a comparison of the results obtained by the various methods.

Example 3

The third example is again one that was originally solved by Hetenyi (page 47), and consists of a weightless beam 10 inches by 8 inches with the loading as shown in Table 6, and resting on an elastic foundation with a constant modulus of 200 lbs. per cubic inch. The beam constants are E = 1.5 x 10^6 psi and I = 426.7 inches.(3)

This problem was solved by the Newmark Method, although only the results have been recorded. Convergence would be slow because of the high value of the spring constant, and it was necessary to use average values of M_a and M_o to speed up the convergence. Using the average values, sufficiently accurate results were obtained after three cycles, and the results are recorded in Table 6.

The solution by the finite difference method is shown in Table 6. The moments at the various subdivisions will be

$$M_B = - kh^2 \left(\tfrac{1}{2}A \right)$$

$$M_C = - kh^2 (A + B) + (150 - 3 = 153) \qquad \text{left}$$

$$M_C = - kh^2 (D + E) + 80 \qquad \text{right}$$

$$M_D = - kh^2 \left(\tfrac{1}{2}E \right) + 5$$

and the general equations are as shown in the solution.

This example was also solved by the finite difference method using the more accurate straight line formulas $\frac{h}{6}(A + 4B + C)$. A comparison of the various results is shown in Table 6. For most purposes sufficient accuracy is obtained by any of the approximate methods, although the finite difference method outlined is the least time consuming.

Example 4

This final example has been added to indicate a method of handling concentrated moments. The sketch shown in Table 7 represents any type of beam resting on an elastic foundation, and with dimensions such as to give values of E = 300,000 kips per sq. ft., and $I = 20$ ft.$^{(4)}$ The beam is symmetrically loaded with two equal concentrated moments—one clockwise, the other counter clockwise. The foundation modulus is 6000 kips per lineal ft., and it is assumed that the foundation can take negative loads. (This would merely mean a reduction in the pressure caused by any direct loads.) Remembering that bending in a beam is a direct result of moment area (angle change), the justification for the 1/2 M in the first equation is recognized. This is an average value of the moment area between A and C. The solution, as well as a comparison with that by a rigorous method using the formulas and tables of Hetenyi's book are shown in Table 7.

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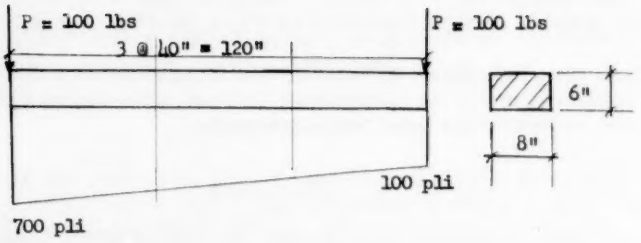


First Cycle:

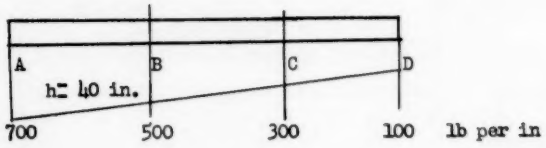
y_a	+	6	+	1	+	3	+	10			10^{-3} in.
q	+	4.20	+	0.50	+	0.90	+	1.00			lbs per in
$\bar{q}$ + P	-	14.25	+	10.10	+	10.50	-	24.05			$\frac{h}{12}$ lbs
V		-14.3		-4.2		+6.4		-17.6			"
M		0	+	14.3	-	18.5	-	12.1			$\frac{h^2}{12}$
M_{c1}		0	-	6.9	-	10.8		0			"
M_{c2}		0	+	4.2	+	10.0	-	12.1			"
M_a		0	-	17.0	-	19.3		0			"
$\bar{\alpha}$		X	+	188	+	209		X			$\frac{h^3}{144EI}$
ϕ		-188		0		+209					"
y	+	188		0		0	+	209			$\frac{h^4}{144EI}$
y'	+	0.928		0		0	+	1.03			inches
q	+	6.51		0		0	+	1.03			lbs per in
$\bar{q}$ + P	-	7.22	+	6.51	+	1.03	-	26.40			$\frac{h}{12}$ lbs
V		- 7.22		-0.71		+0.32		-26.08			"
M		0	-	7.22	-	7.93	-	7.61			$\frac{h^2}{12}$
M_{c1}		0	-	10.25	-	16.08	-	0			"
M_{c2}		0	+	2.66	+	6.32	+	7.61			"
M_o		0	-	14.81	-	17.69		0			"

TABLE 1A : Solution of Problem 1

Second Cycle :

M_a	0	-	15.0	-	18.0		0		do.
$\overline{\kappa}$	X	+	168	+	195		X		
ϕ	-168		0		+195				
y	+ 168		0		0	+	195		
y'	+ 0.880		0		0	+	0.963		
q	+6.16		0		0	+	0.963		
$\overline{q}$ + P	→ 8.44	+	6.16	+	0.96	-	26.63		
V	-8.44		-2.28		-1.32		-27.95		
M	0	-	8.44	-	10.72	-	12.04		
N_{c1}	0	-	11.00	-	17.2		0		
N_{c2}	0	+	4.22	+	10.0	+	12.04		
N_o	0	-	15.2	-	17.9		0		
q	+ 6.16		0		0	+	0.96		lbs per in
q_{c1}	- 3.34	+	0.60	+	2.14	+	1.31		"
q_{c2}	+ 1.08	+	0.09	-	0.36	-	0.36		"
q_o	+ 3.90	+	0.69	+	1.78	+	1.91		"

TABLE 1B : Solution of Problem 1 (cont)



Uniform deflection : ("D")

y_a	+	1	+	1	+	1	+	1	10^{-2}	inches
q	+	7	+	5	+	3	+	1	lbs	per in
$\overline{q}$	+	38	+	60	+	36	+	10	$\frac{h}{12}$	lbs
V		+38		+98		+134		+144		"
M		0	+	38	+	136	+	270	$\frac{h^2}{12}$	lb in

Uniform Rotation : ("C")

y_a	+	3	+	1	-	1	-	3	do
q	+	21	+	5	-	3	-	3	
$\overline{q}$	+	90	+	68	-	28	-	22	
V		+90		+158		+130		+108	
M		0	+	90	+	248	+	378	

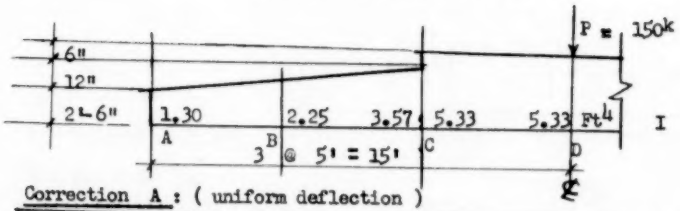
Correction A : ("C" - 0.76 "D")

q	+	15.75	+	1.25	-	5.25	-	3.75	do
V		--		--		--		0	
M		0	+	61.5	+	146.0	+	175.5	

Correction B : ("C" - 1.40 "D")

q	+	11.20	-	2.00	-	7.20	-	0.40	do
V		--		--		--		-93.6	
M		0	+	36.8	+	57.6		0	

TABLE 2 : Correction Configurations --Problem 1



Correction A : (uniform deflection)

y_a	+	1		+	1		+	1		+	1	10^{-2} ft
q	+	3		+	3		+	3		+	3	kips per ft
$\bar{q}$	+	18		+	36		+	36		+	36	kips x $\frac{h}{12}$
V			+18			+54			+90		+216	"
M		0		+	18		+	72		+	162	$\frac{h^2}{12}$ kip ft

First Cycle :

y_a	+	1		+	2		+	3		+	3	10^{-2} ft
q	+	3		+	6		+	9		+	9	kips per ft
$\bar{q}$ + P	+	24		+	72		+	105		-	252	$\frac{h}{12}$ kips
V			+24			+96			+201		+150	"
M		0		+	24		+	120		+	321	$\frac{h^2}{12}$ kip ft
M_c		0		-	12.5		-	50		-	112	"
M_a		0		+	11.5		+	70		+	209	"
α		0		-	5.12	-19.6		-13.1		-	39.2	$\frac{h^2}{12\ EI}$
$\bar{\alpha}$		X		-	70.8		-	260		-	418	$\frac{h^3}{144\ EI}$
ϕ			+540			+469			+209			"
y		0		+	540		+	1009		+	1218	$\frac{h^4}{144E\ I}$
y'		0		+	0.540		+	1.009		+	1.218	10^{-2} ft
q		0		+	1.62		+	3.03		+	3.65	do
$\bar{q}$ + P	+	3.34		+	19.2		+	35.6		-	317.4	
V			+3.3			+22.5			+58.1		- 201	
M		0		+	3.3		+	25.8		+	83.9	
M_c		0		+	16.7		+	67.0		+	151.0	
M_o		0		+	20.0		+	92.8		+	234.9	

TABLE 3 A: Solution of Problem 2

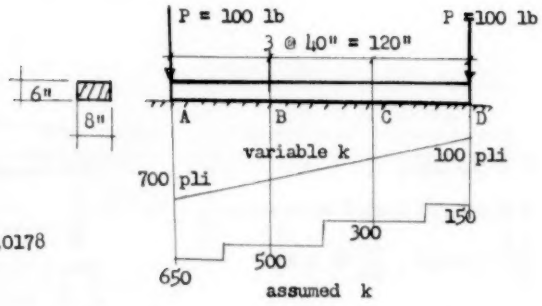
Second Cycle :

M_a	0	+	20.0	+	92.8	+	235	do
M_o	0	+	18.3	+	88.4	+	229	

Third Cycle :

M_a	0	+	18.7	+	89.2	+	229.6	do
M_o	0	+	18.7	+	89.4	+	230.4	
q	0	+	1.90	+	3.46	+	4.16	kips per ft
q_c	+ 2.46	+	2.46	+	2.46	+	2.46	"
q_o	+ 2.46	+	4.36	+	5.92	+	6.62	"

TABLE 3B : Solution of Problem 2 (cont.)



k = varies; h = 40"

$E = 2.5 \times 10^6$ psi

$I = 144$ in^4

$\frac{kh^4}{EI} = 0.00710$; $\frac{Ph^3}{EI} = 0.0178$

Equations :

$$A - 2B + C = -\frac{kh^4}{EI}\left(\tfrac{1}{2}A\right) + \frac{Ph^3}{EI}$$

$$B - 2C + D = -\left(A + B\right) + \frac{2Ph^3}{EI}$$

$$B - 2C + D = -\left(\tfrac{1}{2}D\right) + \frac{Ph^3}{EI}$$

$$3.25A + 5B + 3C + .75D = \frac{2P}{h} \times 10^{-2}$$

Solution :

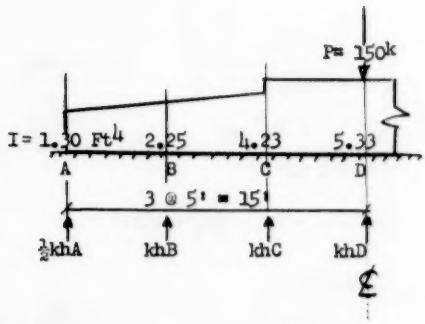
A	B	C	D	10^{-2}
3.31	-2	1		1.78
4.61	4.55	-2	1	3.56
	1	-2	1.53	1.78
3.25	5	3	.75	5
.478	.138	.504	1.74	

Deflection x 10^{-2} inches

Foundation Pressure: (lbs per inch)

A	B	C	D	
3.35	0.69	1.51	1.74	Finite Diff. (above)
3.90	0.69	1.78	1.91	Newmark Method (Malter)
3.54	0.73	1.70	1.79	Popov
3.62	0.57	1.65	1.89	Hetenyi

TABLE 4 : Solution of Problem 1



$k = 300$ klf ; $h = 5$ ft

$E = 432{,}000$ ksf; I = varies

$\frac{kh^4}{E} = 0.434$; $\frac{3}{2kh} = 1 \times 10^{-3}$

Equations :

$$A - 2B + C = -\frac{0.434}{2.25}\left(\tfrac{1}{2}A\right)$$

$$B - 2C + D = -\frac{0.434}{4.23}\left(A + B\right)$$

$$C - 2D + C = -\frac{0.434}{5.33}\left(1.5A + 2B + C\right)$$

$$A + 4B + 2C + 2D = \frac{3P}{2kh} \quad \text{(Simpsons Rule)}$$

Solution :

A	B	C	D	$Px10^{-3}$
1.096	-2	1		0
.102	1.102	-2	1	0
.122	.163	2.081	-2	0
1	4	2	2	1
.0590	.0975	.130	.147	

Deflection x 10^{-3}P ft.

Foundation Pressures : (kips per ft.)

A	B	C	D	
2.66	4.38	5.85	6.62	Finite Diff. (above)
2.5	4.4	5.9	6.7	Newmark Method (Walter)
2.7	4.5	5.9	6.5	Popov

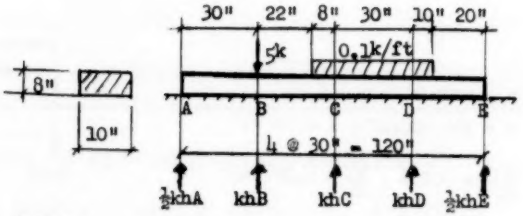
TABLE 5 : Solution of Problem 2

$k = 2$ k/in; $h = 30$ in

$E = 1.5 \times 10^3$ ksi

$I = 427$ in^4

$\frac{kh^4}{EI} = 2.53$; $\frac{h^2}{EI} = 141 \times 10^{-3}$



Equations :

$$A - 2B + C = -\frac{kh^4}{EI}\left(\tfrac{1}{2}A\right)$$

$$B - 2C + D = -\left(A + B\right) + \frac{h^2}{EI}\left(153\right)$$

$$B - 2C + D = -\left(D + E\right) + \left(80\right)$$

$$C - 2D + E = -\left(\tfrac{1}{2}E\right) + \left(5\right)$$

$$\tfrac{1}{2}A + B + C + D + \tfrac{1}{2}E = \frac{1}{kh}\left(5 + 4.8 = 9.8\right)$$

Solution :

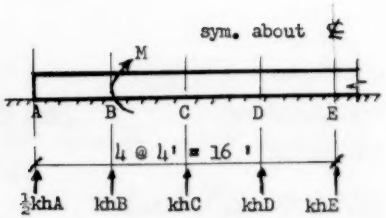
A	B	C	D	E	10^{-1}
2.27	-2	1			0
2.53	3.53	-2	1		2.15
	1	-2	3.53	2.53	1.12
		1	-2	2.27	.07
1	2	2	2	1	3.27

Deflections : (inches)

A	B	C	D	E	
.029	.058	.049	.035	.011	Finite Diff. (above)
.021	.056	.052	.034	.005	Newmark Method (Malter)
.029	.057	.053	.033	.007	Popov (interpolated)
.030		.052		.006	Hetenyi (rigorous)
.023	.061	.053	.036	.006	Finite Diff.* (using $\frac{h}{6}(A + 4B + C)$)

* Solution obtained by Finite Diff. method using straight line variation for foundation pressure.

TABLE 6 : Solution of Problem 3



$k = 6000$ klf; $h = 4$ Ft

$E = 300{,}000$ksf ; $I = 20$ Ft^4

$\frac{kh^4}{EI} = 0.256$; $\frac{h^2}{EI} = 0.267 \times 10^{-5}$

Equations :

$$A - 2B + C = -\frac{kh^4}{EI}\left(\tfrac{1}{2}A\right) - \frac{h^2}{EI}\left(\tfrac{1}{2}M\right)$$ (average M from A to C)

$$B - 2C + D = -\left(A + B\right) - \left(M\right)$$

$$C - 2D + E = -\left(\tfrac{3}{2}A + 2B + C\right) - \left(M\right)$$

$$D - 2E + D = -\left(2A + 3B + 2C + D\right) - \left(M\right)$$

$$\tfrac{1}{2}A + B + C + D + \tfrac{1}{2}E = 0$$

Solution :

A	B	C	D	E	$-\frac{h^2M}{EI}$
1.128	-2	1			.5
.256	1.256	-2	1		1
.384	.512	1.256	-2	1	1
.512	.768	.512	2.256	-2	1
.5	1	1	1	.5	0
1.565	.485	-.292	-.618	-.714	

Deflections (ft) ($\frac{-h^2M}{EI}$)

Foundation Pressure (kips per ft.) (10^{-5} M)

A	B	C	D	E	
-.418	-.130	.078	.171	.190	Finite Diff. (above)
-.432	-.138	.078	.172	.188	"Rigorous Method" (Malter)*

* (using Hetenyi)

TABLE 7 : Solution of Problem 4

Paper 1563 March, 1958 ST 2

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STRUCTURAL DIVISION
Proceedings of the American Society of Civil Engineers

EARTHQUAKE RESPONSE OF ELEVATED TANKS AND VESSELS

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(Proc. Paper 1563)

SYNOPSIS

Recent western United States earthquakes, while not being as severe as can be expected to occur, have provided tests of many structures specifically designed to resist earthquake forces.

A method of analysis is discussed wherein minor damage to elevated water tank towers and refinery vessels can be satisfactorily explained.

INTRODUCTION

Recent western United States earthquakes have provided data[1,2] on damage to earthquake resistive elevated water storage tanks and refinery type structures. This damage, while being slight, still indicates some weaknesses in our present methods of analysis.

Since these particular types of structures can be treated relatively easily as dynamic structures, the authors have chosen them as examples to be mathematically subjected to actual loadings as indicated by instrumental records of recent strong shocks including those in which they suffered damage.

Earthquake Spectra

Research using an electric analog computer to analyze strong motion accelerogram records has produced spectra that can be used to predict stresses in structures when subjected to actual earthquake motion.[3] Briefly, the earthquake spectrum is a plot of the maximum response of a simple oscillator versus the period of the oscillator when subjected to actual ground motion at

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a particular location in a particular earthquake. Various percentages of critical damping of the oscillator can be introduced in the spectrum and, as shown in Figures A and B, relatively small amounts reduce the structural response sharply. Figures A and B are acceleration spectra and show the maximum acceleration with respect to gravity versus the oscillator period for various percentages of critical damping. These spectra are plotted for one direction only at each location although two are generally made for two directions 90° apart.

Damping

Tests[5,6] have shown that percentages of critical viscous damping for various types of construction are approximately as follows:

Type	Per Cent of Critical Damping (Viscous)
Elevated water tank towers (5)	1 - 5
Self-supporting welded steel stacks (6)	
Unlined	0.5 - 1.0
Lined with gunite	1 - 1.5

For elevated tank towers, the damping was found to be decreased by tightening rods and/or strengthening the towers.[5]

Welded steel stacks apparently have smaller damping than riveted types and presumably the use of fire brick lining would increase the damping, but there is a lack of data to substantiate this.[6]

Elevated Water Tank Towers

Experience with elevated tank towers and refinery type structures in the California Earthquake of July 21, 1952,[1] indicates that the type of damage exhibited can be explained when it is realized that these structures have relatively low percentages of critical damping and practically no secondary lateral force resisting elements.

Five of eleven elevated water tanks, which had been designed for earthquake, suffered stretching of their anchor bolts in the Kern County area. Two showed evidences of slightly stretched rods. Slight anchor bolt stretching was also noted on one earthquake resistive tank in Eureka after the earthquake of December 21, 1954.

All of the above tanks have four legs and are generally of welded construction with very thick base plates. The possibility of prying action, due to column bending, contributing to the anchor bolt stretch should not be overlooked. These tanks are used for fire protection service and all were probably full at the time of the shock.

For example, a 100,000 gallon tank on a 100' tower located at the Maricopa Seed Farms, about 15 miles east of Taft (Figure C), suffered stretched anchor bolts and bracing rods. Lateral force design was on the

basis of Pacific Fire Rating Bureau recommendations(7) using 10% gravity and an allowable rod stress of 18,000 pounds per square inch with no increase for lateral forces. Tower design was welded tubular columns and round steel bracing rods with upset end and turnbuckles.

Figure D is a schematic elevation of this tank and tower showing member sizes. Soil conditions are poor and piling was used. Reinforced concrete struts were used around the perimeter of the base and diagonally.

This structure, for purposes of analysis, may be considered as a single mass concentrated at the center of gravity of the tank and supported by a weightless elastic frame.

For this type of structure, one degree of freedom can be assumed and the free period computed from: (Refer to Page 112 of reference 5)

$$T = 2\pi\sqrt{\frac{\delta_{st}}{g}} \qquad \text{I}$$

Where: T = Free period in seconds.

δ_{st} = Deflection in feet, of the top of the tower under a static horizontal force equal to its own weight.

g = Acceleration of gravity in ft. per sec.^{2}

If the rods are assumed to act in tension only, the computed period T = 1.4 seconds. However, pretension in the rods will reduce this period somewhat. Secondary bending in the columns will also decrease this period as all joints were assumed hinged in the above computation. Poor soil conditions should lengthen the period.(5)

Considering these factors as well as comparing with actual tests on similar tanks a period T = 1.25 seconds will be assumed as being more nearly the actual case.

Comparing this tank with similar tanks which have been tested,(5) damping of 2% of critical will be used.

Since the definition of the spectrum in the maximum response of a single degree of freedom structure, we can now estimate the maximum acceleration of this tank, with respect to the ground, by using the acceleration spectra (Figures A and B). Unfortunately no accelerometer was located closer than Taft (Figure C) and no actual spectra are available for the Maricopa Seed Farms area.

Using a period T = 1.25 sec. and n = 0.02, an acceleration of approximately 0.30 G can be picked from Figure B, and approximately 0.12 from Figure A. Assuming that the severity of ground motion at the epicenter of the July 21, 1952, shock was somewhat less than at El Centro in 1940, and considering the location of the tank with respect to Taft and the epicenter, a value of 0.20 G will be used.

Analyzing the tower for a lateral force of 20% of the weight of the tank and contents results in rod stresses from 34,000 to 38,000 pounds per square inch and maximum column stress of approximately 21,200 pounds per square inch. The computed rod stresses are in the neighborhood of the elastic limit and could account for some small amount of permanent elongation. Relative inward movement of the column bases during the earthquake also contributed to the slack rods in the lower panel. Considering their relatively large safety factor, the columns should not and did not show signs of distress due to

compression, but some paint flaking at welded joints where plate thicknesses change could be attributed to secondary bending.

However, the maximum anchor bolt stress, due to forces in a diagonal direction, is about 46,000 pounds per square inch on the gross area which is well above the yield point and could explain the permanent stretching of these bolts. Maximum total elongation including that caused by aftershocks as late as January 12, 1954, is on the order of 1". The column to bolt details were also deformed.

At the time of the El Centro Earthquake of April 18, 1940, there were only two tanks in that area which were designed to resist shock. These tanks were relatively large (250,000 gallons) with six legs. One located in El Centro is on an 80 foot tower and was originally designed for wind only and was subsequently reinforced for earthquake prior to 1940. The other tank, in Brawley, is on a 120 foot tower and was originally designed for a 10% G lateral force. Both tanks were designed in accordance with the recommendations of the Board of Fire Underwriters of the Pacific(7) and suffered no damage. There tanks are used for city water supply and were probably full at the time of the shock.

Using methods outlined above, a computed period T = 1.20 sec. was arrived at for the El Centro tank. There are few data on damping for this size and type of construction. However, due to the riveted construction, a value higher than for a welded type will be used. A value of 5 per cent of critical will be assumed.

Using an average value from both components of the El Centro spectra, a value of approximately 0.20 gravity can be obtained and computed stresses for the rods are 27,700 to 28,500 psi. Anchor bolt stresses are approximately 50,000 psi on the gross section.

The absence of damage can be explained when considering the above stresses by the fact that probably more secondary unconsidered resisting elements are present in a six-leg than in a four-leg tower. Also, the base details on the El Centro tank were flexible when compared with the rigid welded type of the Kern County tanks.

Prying action of the column base plate, due to column bending, would not be as effective on the El Centro tank due to the bolt being in the center of the plate.

The results indicate that moderate allowable rod stresses of 18,000 psi with no seismic increase are desirable.(7) A design based on the 20,000 pounds per square inch plus a 1/3 increase as presently allowed by many codes would have resulted in excessive rod stretching. Anchor bolts should be proportioned so as to keep stresses within the elastic limit when considering the probable actual lateral loads.

An examination of typical spectra discloses that shorter towers with relatively shorter periods will be subjected to higher accelerations in the same location and during the same shock.

This was recognized many years ago and recommendations(7) as early as 1935 were as follows:

Height of Tower on Firm Ground	Horizontal Load as Fraction of Gravity
75 feet or over	0. 10
50 feet to 75 feet	0. 12
Under 50 feet	0. 15

It is important to note that a dynamic analysis supports the observed performance.

Refinery Vessels

Damage to tall slender vessels located at the Paloma Cycling Plant (Figure C) in the California Earthquake of July 21, 1952, was as shown in Figure E.

Data on various vessels were compiled[8] and are shown in Figure F. The measured bolt stretch as shown in column 16 (Figure F) is the average maximum which usually occurred in the northeast quadrant. Soil conditions are only fair at this location. Spread footings were used for all vessels. Figure E shows details of a typical vessel.

For purposes of the following analysis, these vessels will be considered as uniform beams (with constant wall thickness) fixed at their bases.

For a continuous uniform beam subjected to forced vibration it has been shown[4] that deflection (y) at any point (with respect to its supports) is given by:

$$y(x,t) = \frac{WL^3}{EI} \sum_{i=1}^{\infty} \frac{\phi_i(x)\, K_i(x)}{(\beta_i L)^4} D_i(t) \qquad \text{II}$$

Where W = Total weight of beam.

L = Length of beam.

E = Young's Modulus.

n = Mode of vibration.

I = Moment of inertia of beam.

$\phi_i(x)$ = Characteristic function depending on end conditions.

β_i = Characteristic number depending on frequency and physical properties of the beam.

$K_i(x)$ = Participation factor = $\frac{1}{L}\int_0^L \phi_i(x)\, dx$

$D_i(t)$ = Dynamic load factor.

Values of $\phi_i(x)$, $\phi_i''(x)$, $\phi_i'''(x)$, β_i and solutions of K_i have been tabulated for uniform beams with various end condition.[9] All of the above terms except the dynamic load factor are functions of the physical and dynamic properties of the beam and are independent of the forced vibration.

For earthquake motion, the dynamic load factor is obtainable for each mode from the acceleration spectrum and represents the effect of actual earthquake ground motion of the vibration of the beam.[2]

Thus, if we know or can estimate the physical and dynamic properties of the structure, then by using the spectrum for any particular location and earthquake, we can estimate what the maximum response would have been at that location in terms of deflections or, by differentiating equation (II) in

terms of:

$$M(x,t) = WL\sum_{i=1}^{\infty}\frac{\phi_i''(x)}{(\beta_i L)^2}\ K_i\ D_i\ (t) \qquad \text{III}$$

Where $M(x,t)$ = Maximum moment at any point.

and $$V(x,t) = W\sum_{i=1}^{\infty}\frac{\phi_i'''(x)}{\beta_i L}\ K_i\ D_i\ (t) \qquad \text{IV}$$

Where $V(x,t)$ = Maximum shear at any point.

The maximum response assumes the worst condition of all modes of vibration being additive and is on the conservative side. Generally, only the first two or three modes need be considered.

Using equation III the maximum base moment will be computed for the vessels, considering the first two modes only. The weight W used will be the operating weight as given in Figure F.

From reference 9, assuming a clamped-free condition (similar to a cantilever beam) the various values for the clamped end or base are as follows:

First Mode		Second Mode	
$(\beta_1 L)^2$	= 3.52	$(\beta_2 L)^2$	= 22.03
ϕ_1''	= 2.0	ϕ_2''	= 2.0
K_1	= 0.79	K_2	= 0.435
and			
$\frac{\phi_1'' K_1}{(\beta_1 L)^2}$	= 0.45	$\frac{\phi_2'' K_2}{(\beta_2 L)^2}$	= 0.04

In order to pick the proper dynamic load factor, $D_i(t)$ from the spectrum, we must know the period T and the per cent of critical damping.

The period T for each mode can be computed from:

$$T_i = \frac{2\pi}{W_i} \qquad \text{VI}$$

Where W_i = angular velocity = $\beta_i^2\sqrt{\frac{EI}{M}}$

M = mass per unit length and β_i can be taken from reference 9.

By use of equation V and VI the periods shown in Figure G have been computed.

Factors which must be evaluated include:

1. Initial tension in the anchor bolts. None was assumed.

2. The cross-sectional area of the bolt to be used; net area, gross area, or something between these two values. Gross areas were used.

3. The amount of the operating liquid to be included in figuring the weight. Operating weights were used in Figure G.

4. The effect of vertical acceleration. The July 21, 1952, shock was characterized by considerable vertical acceleration probably due to the combination of vertical and horizontal faulting. This was neglected.

5. Damping. This undoubtedly varies among the various vessels due to differences in contents, insulation and foundations. A value of 2% of critical for all was used to arrive at the dynamic load factor in Figure G.

6. Effect of impact after initial stretch of anchor bolts. This was neglected.

Again we are faced with the problem of not having spectra available for the Paloma area. The dynamic load factors $D_i(t)$ shown in Figure G were arrived at by assuming the motion at Paloma to be represented by roughly the average of Taft (Figure A) and two thirds of El Centro (Figure B).

However, the actual magnitude of $D_i(t)$ is not as important here as are its comparative values for the various vessels.

Combining the various factors as indicated by equation III gives the values of base moments M shown in Figure G.

Figure H is a plot of observed average maximum bolt stretch versus computed stress using the base moment M and the gross area of the bolts. The dashed line in Figure H represents an approximately average value of the various points and indicates only minor stretching should be expected if the computed stresses are held in the neighborhood of the elastic limit.

For computed stresses above the elastic limit, the observed stretch increases sharply.

Considering the assumptions made, the relation between the observed bolt stretch and the computed stress is striking and provides ample evidence to warrant dynamic analysis when designing similar structures instead of using the same static factor for all. Allowable stresses in this analysis can approach the yield point when using the maximum response expected because all modes would not peak together. Maximum deflections, moments and shears at any location along a uniform beam with various end conditions can be computed using this method.

Many additional records of strong shocks, particularly on various types of soils, are necessary in order to provide the engineer with spectra which can be assumed as reasonably maxima to be expected for any particular location.

SUMMARY AND CONCLUSIONS

1. Dynamic considerations will undoubtedly result in a more economical use of materials and members in tall structures by considering their action under actual predicted ground motions but present dynamic methods, even when applied to very simple structures, are extremely complicated for the average design office. Over simplification of dynamic analysis when being applied to buildings may result in dangerous errors due to the number of necessary assumptions which must be made.

2. Realistic lateral force factors and allowable stresses for buildings based

on observed damage should not be discarded in favor of a pure dynamic analysis until more data are available.

3. Relatively simple structures, such as elevated tank towers, refinery vessels, stacks and simple framed towers can be investigated dynamically using methods shown in this paper to be predictive of actual damage.

ACKNOWLEDGMENTS

H. M. Engle, Consulting Structural Engineer, and K. V. Steinbrugge, Structural Engineer, Pacific Fire Rating Bureau, by virtue of their experience in seismic design and investigation of earthquake damage have offered valuable suggestions in the preparation of this paper.

Professor C. Martin Duke, University of California at Los Angeles, contributed by reading the manuscript and offering valuable suggestions.

V. R. Bush and J. D. B. Allison, Structural Engineers for the Pacific Fire Rating Bureau, have also contributed by checking data and computations.

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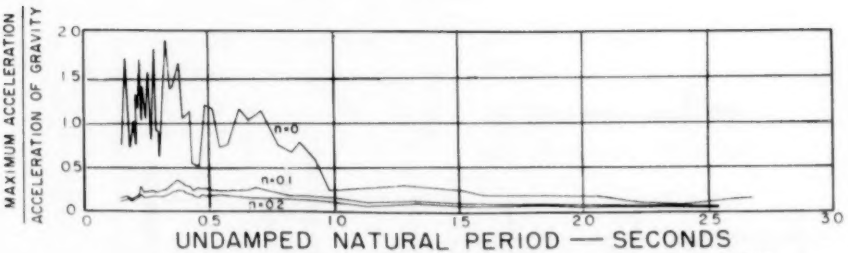


Figure A. Acceleration spectra for Taft, California, earthquake of July 21, 1952 Component N21E

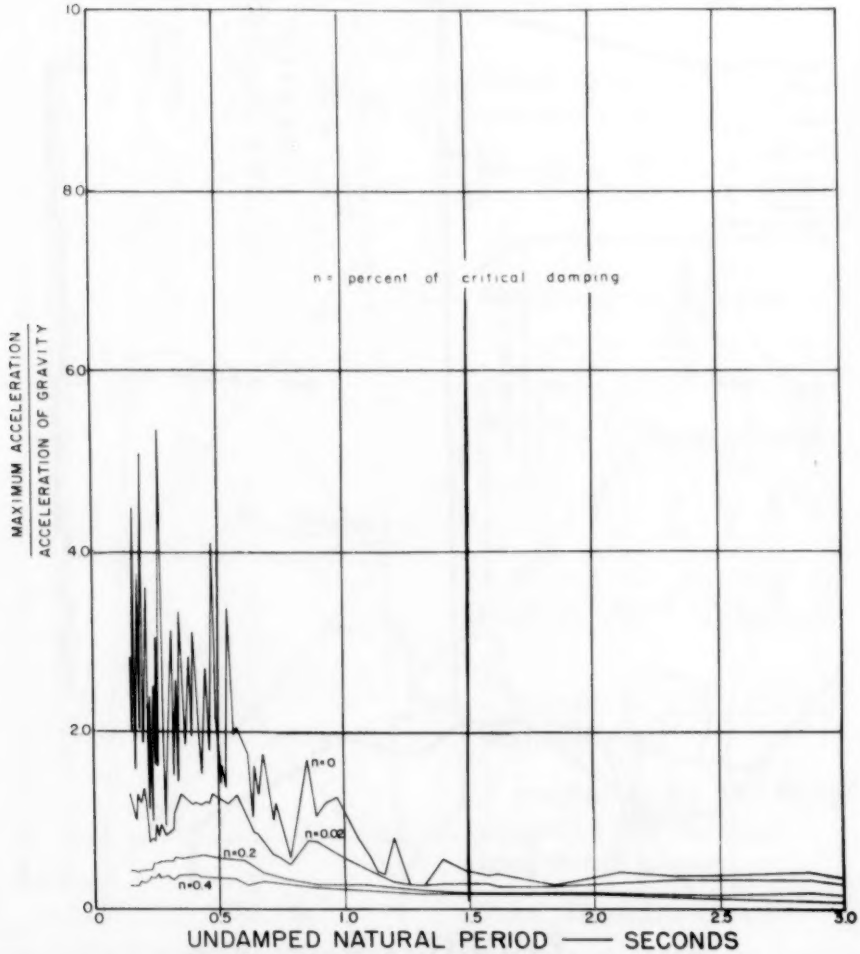


Figure B. Acceleration spectra for El Centro, California; earthquake of May 18, 1940. Component N-S.

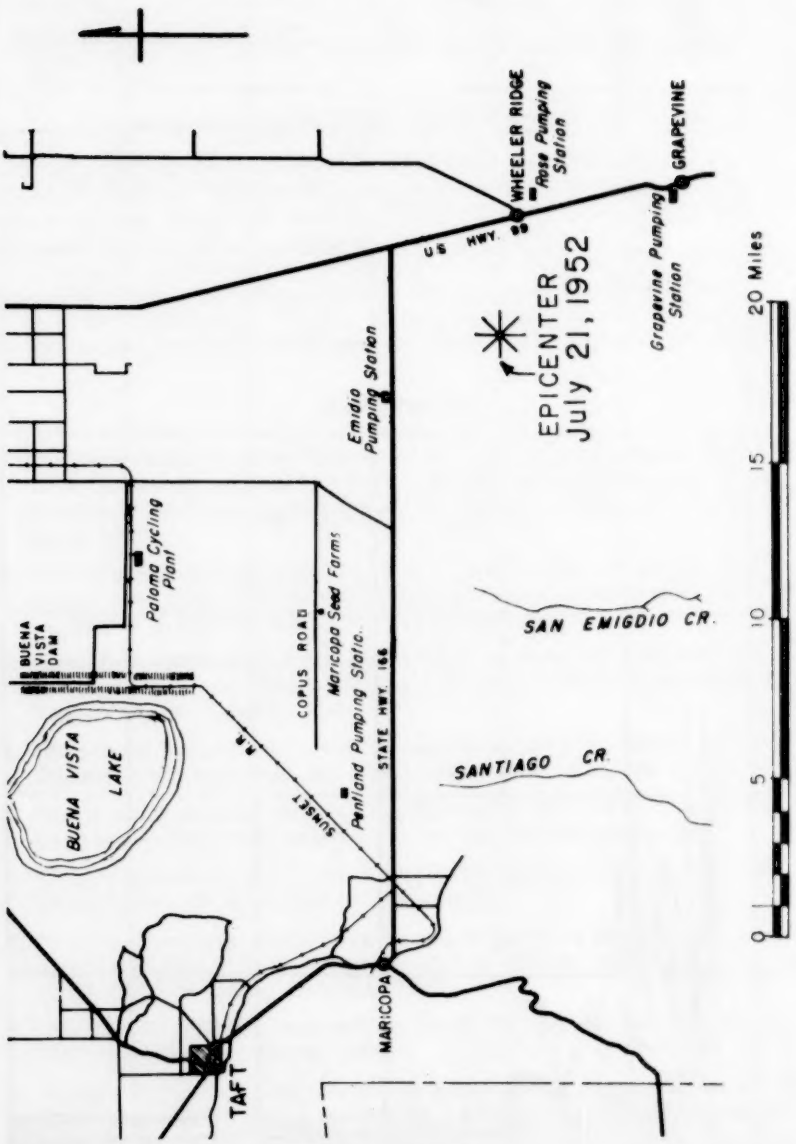
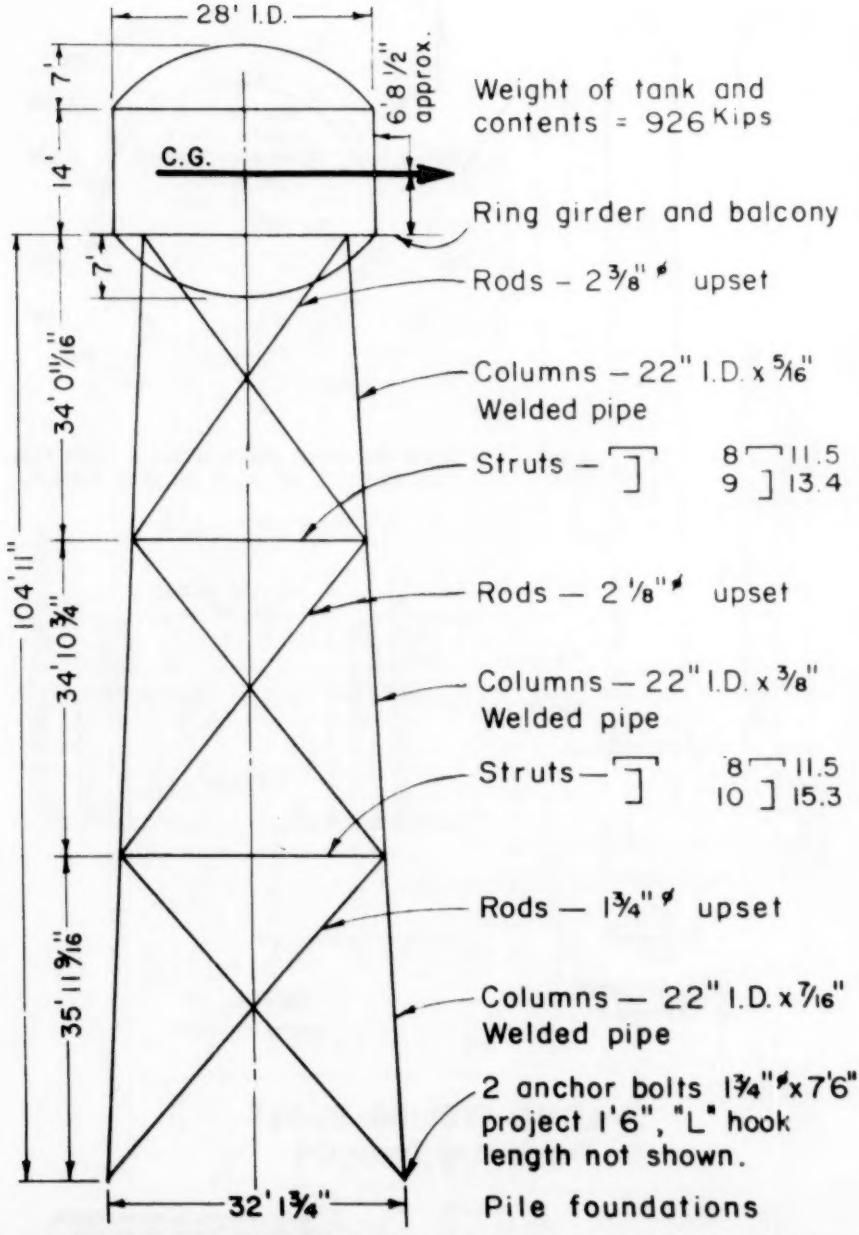
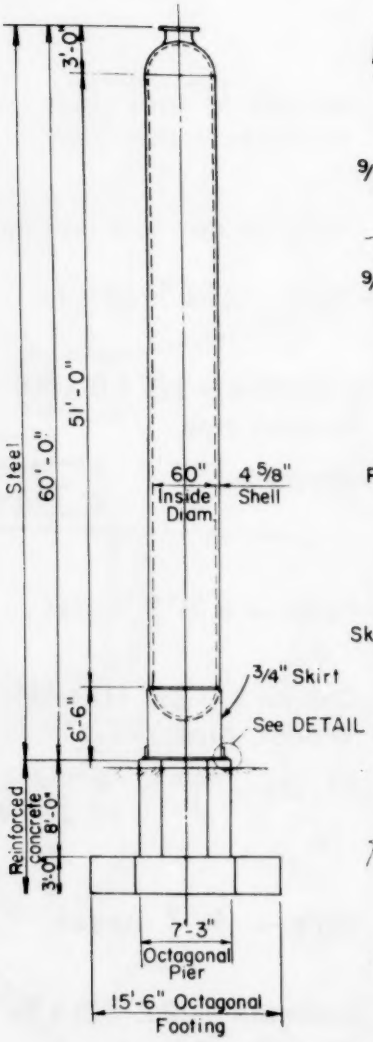

WHEELER RIDGE
Rose Pumping Station
GRAPEVINE
Grapevine Pumping Station
US HWY. 99
EPICENTER
July 21, 1952
Emidio Pumping Station
Paloma Cycling Plant
BUENA VISTA DAM
COPUS ROAD
Maricopa Seed Farms
Panland Pumping Station
STATE HWY. 166
SUNSET
BUENA VISTA LAKE
SAN EMIGDIO CR.
SANTIAGO CR.
MARICOPA
TAFT
0
5
10
15
20 Miles

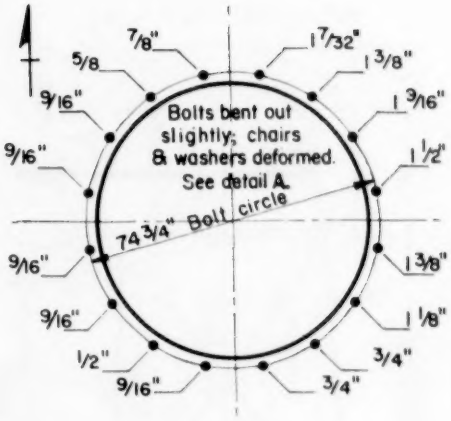
FIGURE C


28' I.D.
7'
14'
7'
6'8 1/2" approx.
C.G.
Weight of tank and contents = 926 Kips
Ring girder and balcony
Rods – 2 3/8" ϕ upset
Columns – 22" I.D. x 5/16" Welded pipe
Struts – 8 11.5 9 13.4
Rods – 2 1/8" ϕ upset
Columns – 22" I.D. x 3/8" Welded pipe
Struts – 8 11.5 10 15.3
Rods – 1 3/4" ϕ upset
Columns – 22" I.D. x 7/16" Welded pipe
2 anchor bolts 1 3/4" ϕ x 7'6" project 1'6", "L" hook length not shown.
Pile foundations
104'11"
34'0 11/16"
34'10 3/4"
35'11 9/16"
32'1 3/4"

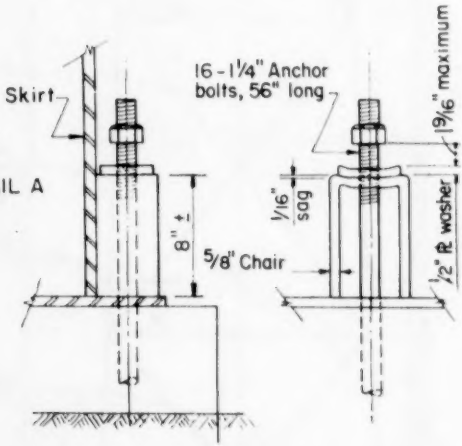
SCHEMATIC ELEVATION OF TANK AND TOWER

FIGURE D





PLAN SHOWING ANCHOR BOLT STRETCH AS RESULT OF JULY 21, 1952 SHOCK



DETAIL A

Welding not shown

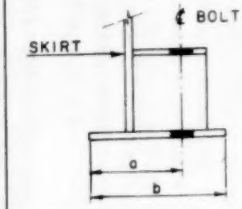
PALOMA CYCLING PLANT
Processing Tower C-I

Figure E

NOTE: DRAWINGS BASED ON GENERAL PETROLEUM "REPORT ON EFFECTS ... OF THE TEHACHAPI-ARVIN EARTHQUAKE".

VESSEL					WEIGHT & CENTER OF GRAVITY (a) EMPTY		OPERATING		ANCHOR BOLTS						
No	Inside Diam Inches	L Height Feet	Approx Shell Thickness Inches	Skirt Thickness Inches	Wt Kips	Height to C.G Feet	W Wt Kips	Height to C.G Feet	No	Size Inches (D.a)	Type (b)	Area Sq Inches (c) Gross	Area Sq Inches (c) Net	Bolt Circle Inches	Measured Bolt Stretch Inches (d)
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
C 1	60	60	4 3/8	3/4	203	30.3	211	30.0	16	1 1/4	L 56	19.62	14.86	74 3/4	1 1/2 - 1 3/4
C 4	66	59	7/16	3/8	52	31.5	62	29.3	12	1 1/4	J 51	14.72	11.15	71 7/8	1/8 - 3/16
C 5	54	68	1/4	3/8	29	35.0	36	33.0	16	1 1/4	J 51	19.62	14.86	59 3/4	1/16 - 1/8
C 6	78	104	1	5/8	141	55.5	160	54.5	16	1 3/4	J 70	38.50	31.70	85 1/2	0 - 1/16
C 7	66	97	1 1/16	5/8	111	49.5	123	48.2	16	1 3/4	J 70	38.50	31.70	73 1/8	0
C 8	36	48	3/8	3/8	16	25.5	18	24.7	8	1 1/2	J 58	14.14	11.24	41 3/4	0 - 1/16
C 9	48	97	7/16	1/2	50	49.4	57	48.0	16	1 3/4	J 72	38.50	31.70	54 1/2	0 - 1/16
C60	78	84	2 1/4	1/2	203	43.3	225	42.0	16	1 3/4	J 45	38.50	31.70	88	5/8 - 3/4
C64	108	80	1	1/2	162	43.3	223	39.0	20	1 1/4	L 53	24.54	18.58	115	1/2 - 5/8
C66	60	105	7/8	1/2	109	49.0	120	46.5	12	2	J 60	37.70	31.80	67 1/2	1/8 - 3/16
C67	42	54	1 1/16	3/8	35	31.0	39	30.0	8	1 1/2	J 45	14.14	11.24	49	3/8 - 1/2
C68	42	94	7/16	1/2	39	50.0	46	47.0	12	2	J 60	37.70	31.80	49	0 - 1/8

BASE RING DIMENSIONS



Bolt ϕ	a	b
1"	5"	3 1/2"
1 1/4"	5 3/4"	3 3/4"
1 1/2"	6 1/4"	3 7/8"
1 3/4"	6 3/4"	4"
2"	7 1/4"	4 1/2"

(a) Height to center of gravity is above base ring.

(b) The letter refers to the shape; the number indicates the over-all length in inches.

(c) The threads are eight thread series.

(d) Average maximum values. See Figure E.

C.G. Center of gravity.

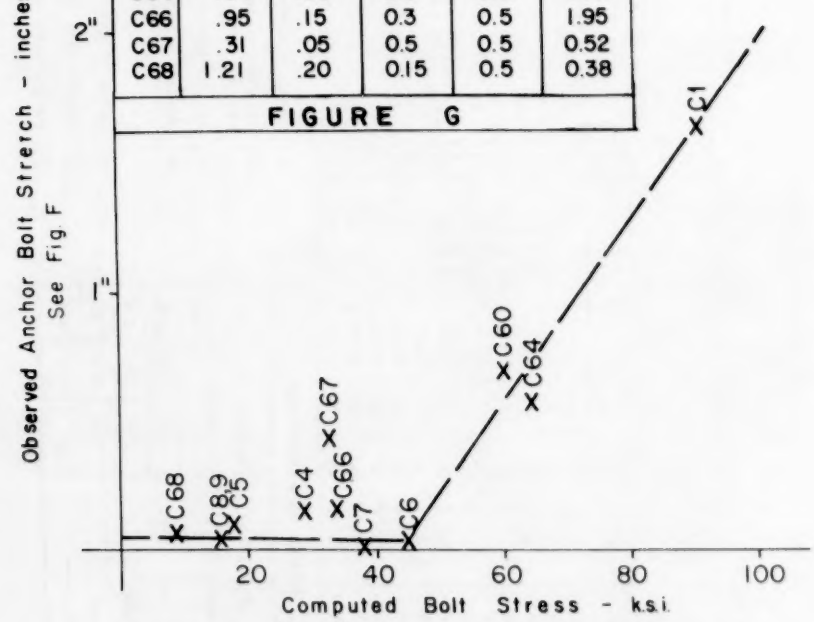
L Length used in Equation III.

W Weight used in Equation III.

FIGURE F FRACTIONATOR COLUMN PROPERTIES

NO.	PERIOD - Ti SECONDS		DYNAMIC LOAD FACTOR - Di(t)		MAXIMUM BASE MOMENT M Foot kips x10³
	First Mode	Second Mode	First Mode	Second Mode	
C 1	.22	.04	0.5	0.5	3.11
C 4	.35	.06	0.4	0.5	0.73
C 5	.71	.05	0.4	0.5	0.49
C 6	.69	.11	0.4	0.5	3.33
C 7	.69	.11	0.4	0.5	2.38
C 8	.38	.06	0.5	0.5	0.21
C 9	1.16	.19	0.25	0.5	0.73
C60	.50	.09	0.5	0.5	4.63
C64	.34	.05	0.5	0.5	4.38
C66	.95	.15	0.3	0.5	1.95
C67	.31	.05	0.5	0.5	0.52
C68	1.21	.20	0.15	0.5	0.38

FIGURE G



Based on $\frac{M}{S} - \frac{W}{A}$ where S = Section modulus of bolt pattern about diameter of bolt circle.

A = total gross bolt area

M = moment from Figure G.

W = operating weight from Figure F.

FIGURE H

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ANALYSIS OF OPEN-SPANDREL ARCHES

A. F. Diwan,[1] A.M. ASCE
(Proc. Paper 1564)

SYNOPSIS

This paper presents a mathematical solution for the open-spandrel arch as a monolithic structure assuming full continuity between the arch rib, the spandrel columns and the deck. The solution is based on the principle of balancing the panel moments by imposing therein a special type of panel distortion which produces moments of a known pattern in the chords of the deformed panel only, and nowhere else in the open-spandrel arch. This makes it now possible to study mathematically the interaction between the deck, the spandrel columns and the arch rib, as well as the effect of introducing expansion joints in the deck, and thus make use of the so called deck participation in the design of this type of arch bridge.

INTRODUCTION

When continuity is considered between the arch rib, the spandrel columns and the deck, the open-spandrel arch bridge becomes a complex system for which the mathematical analysis by the classical methods is so complicated that it has seldom been undertaken. However, the existence of interaction between the various parts of the open-spandrel arch have long been recognized and the near impossibility of the classical methods of analysis has therefore led to extensive experimental studies, especially those by Professors Finlay,(1) Wilson(2,3) and Newmark.(4) These experiments were almost always limited to the determination of the influence lines for the fixed end reactions at the springings and for the moment and thrust at the arch crown, and thus did not give full information as to the quantitative distribution of stresses between the deck, the spandrel columns and the arch rib.

Note: Discussion open until August 1, 1958. A postponement of this closing date can be obtained by writing to the ASCE Manager of Technical Publications. Paper 1564 is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 84, No. ST 2, March, 1958.

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The object of this paper is to present a simple mathematical analysis for the open-spandrel arch system which enables the interaction between the deck, the spandrel columns and the arch rib to be taken into account when designing such systems. In principle the method is based on the concept of balancing the panel moments described recently by the author for the analysis of the Vierendeel girder,(6) in which the solution is obtained by successive operations of joint rotation and special panel displacements producing moments in the chords of the distorted panel only without affecting the stresses or equilibrium of other panels.

Since the open-spandrel arch, whether totally fixed at the springing, or continuous over elastic piers, is externally statically indeterminate, (as opposed to the externally determinate Vierendeel girder), it becomes necessary to obtain first the influence lines for the components of the fixed end reaction, and from these, the stress analysis due to any particular case of loading, temperature changes or displacements at supports, and also the influence line for any stress function at internal sections, can be readily obtained. The application of the method is illustrated by reference to the reinforced concrete models of nine panel open-spandrel arches used previously in experimental work, some of which were also solved mathematically by the method of "equivalent virtual panels" developed by the author and presented later by L. Beaufoy.(7) The agreement between the results given here and those previously obtained, both mathematically and experimentally, is remarkable.

I. Description of the Method of Analysis

1.1 - The Special Panel Displacement

Let us first consider a single isolated panel ABCD (Fig. 1-a) and assume that joint C is displaced vertically to C_1, while A and B are kept fixed. If no axial deformation is allowed joint D will have to move in a direction normal to AD to the new position D_1 such that:

$$D D' = C C_1 = \alpha s$$

$$D_1 D' = \alpha b$$

where α is any arbitrary constant.

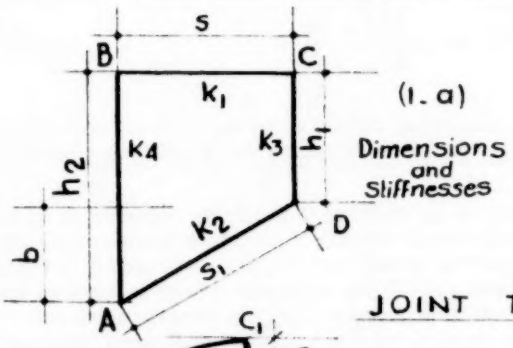
If no rotation is allowed at joints C and D (as well as A and B), then the B.M.D. produced in the panel will be as shown in Fig. (1-b).

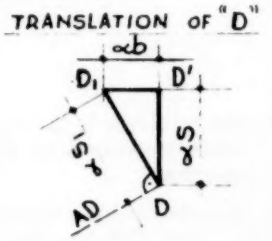
If, on the other hand, a rotation $\phi = (\alpha b/h_1)$ is produced at the two joints C and D only, with no joint translation, we shall get a B.M.D. similar to that in Fig. (1-C).

If these two sets of panel distortion, namely the joint translations in Fig. (1-b) and the joint rotations in Fig. (1-C) are both imposed at the same time, a special type of panel displacement similar to that in Fig. (1-d) is obtained. The B.M.D. thus produced in the panel, which is also shown in Fig. (1-d), is very interesting. By considering this diagram it is seen that:

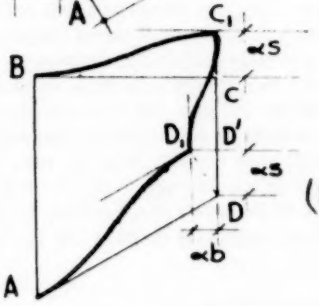
1. No bending moment is produced in the verticals AB and CD.
2. The points of zero moment G_1 and G2 in the top and bottom chords respectively lie on a vertical line passing through the centre of area of the panel, irrespective of the moments of inertia of the panel members.

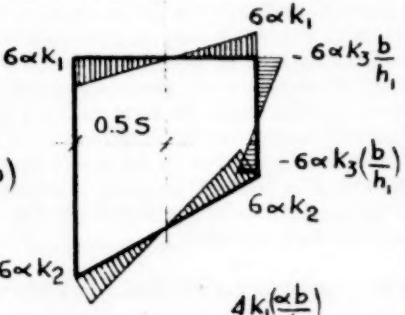
FIG. (1) SPECIAL PANEL DISPLACEMENT.

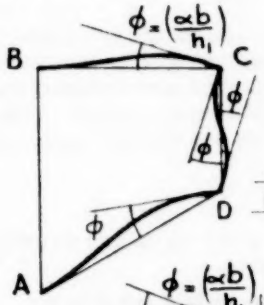




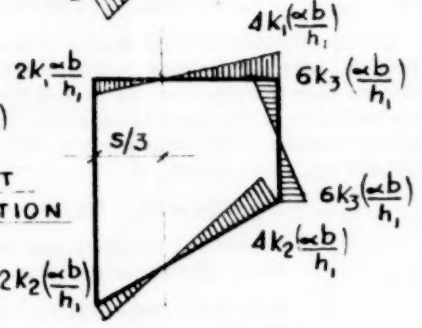
JOINT TRANSLATION

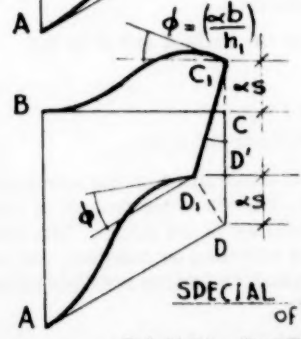




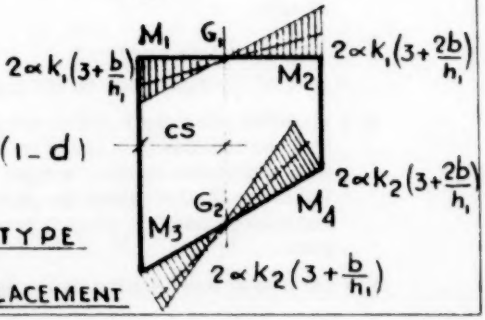


JOINT ROTATION





SPECIAL TYPE of PANEL DISPLACEMENT



If CS = distance of line $G_1 G_2$ through the centroid from the longer vertical side AB; then:

$$c = \frac{2h_1 + h_2}{3(h_1 + h_2)} \qquad (1)$$

3. The moments in the top and bottom chords along any vertical section are directly proportional to the stiffness values K_1 and K_2 of these chords respectively.

In fact, the final displacement of the vertical C D may be regarded as an upward translation equal to (α s) plus a rotation of the whole member as a rigid body, through an angle $\phi = (\alpha b/h_1)$ about point C_1, the new position to which C has moved. This is the reason why CD has remained free from bending moments. If now panel ABCD is one of a group of panels forming an open-spandrel arch system it can be seen that the special type of panel displacement in Fig. (1-d) may only be imposed on the panel when the system is externally free or simply supported at the springing. In this case no moments will be produced any where except in the chords BC and AD of the deformed panel ABCD, since the total relative displacement of all panels to the right of CD with respect to those to the left of AB will also consist of a vertical translation plus a rotation of the whole group of panels as a rigid body about C_1. Hence, it is possible to produce in any panel of the system a bending moment diagram similar to the pattern in Fig. (1-d) by imposing therein the necessary special displacement.

1.2 - Condition for the Static Equilibrium in the Panel

Let us consider now the equilibrium of the two chords of any internal (or external) panel in the open-spandrel arch as shown in Fig. (2). For convenience we may imagine that the rest of the open-spandrel arch including the two verticals of the panel is substituted by its action on the chords. This action consists of the moments M_1, M_2, M_3 and M_4 and the horizontal and vertical forces as shown. If:

$\Sigma M_C = M_1 + M_2 + M_3 + M_4$

= sum of the moments applied by the joints on the two chord members. clockwise moments are positive.

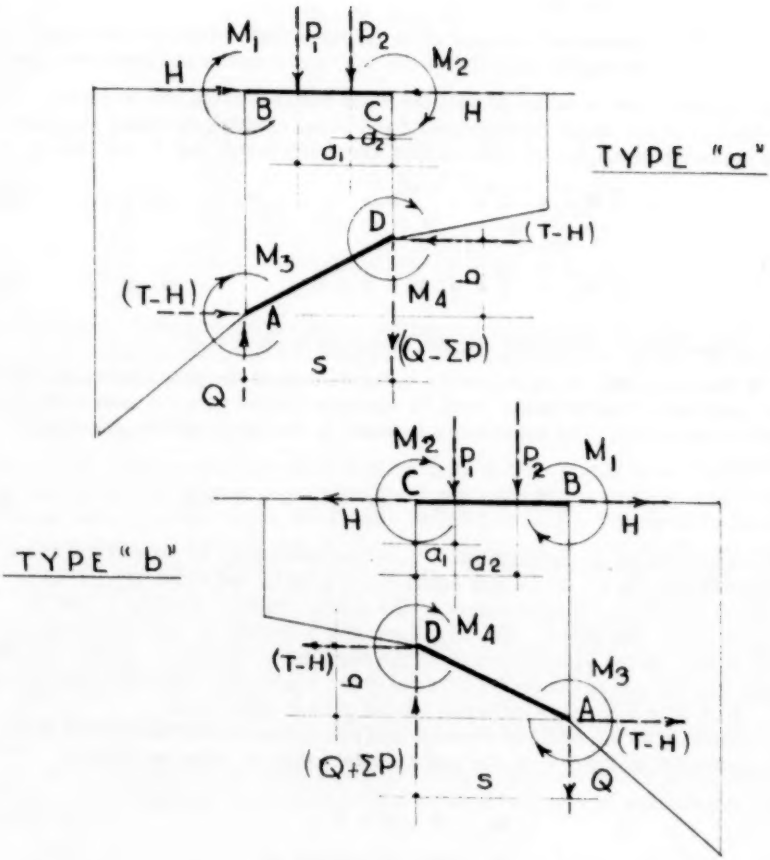
Q = sum of the two shearing forces at the ends A and B of the chords.

$P = P_1 + P_2 + \ldots\ldots\ldots$

= sum of vertical loads on the top chord BC if any.

Q' = modified shearing force in the panel obtained on the assumption that the intermediate loads on the chords of the panels, if any, are substituted by their simple actions at the joints. The shearing force Q' throughout the panel will then be constant, and is considered positive when it produces clockwise rotation in the panel.

H = horizontal force acting on the top joint of the panel to the side of



STATIC EQUILIBRIUM OF PANELS

FIGURE (2)

the longer vertical members. This will be the joint to the left in type (a) and to the right in type (b). H = sum of the shears in all the spandrel columns, including the vertical member in the panel itself, to the side in which the near support exists; i.e. all spandrel, columns to the left in type (a) and to the right in type (b).

T = horizontal component of the fixed end reaction at the near springing (that to the left for type a and to the right for type b).

Both T and H are considered positive when they act from left to right.

Then, for the static equilibrium of the panel chords, by taking the sum of the moments of all forces and couples about any point, say D, we get:

$$\Sigma M_C + Q s - \Sigma P.a - b (T - H) = 0 \qquad (2)$$

or

$$\Sigma M_C + Q' s - b (T - H) = 0 \qquad (3)$$

1.3 - Balancing Coefficients for the Moments in the Panel

If the sum ΣM_C of the moments in the chords of the panel does not satisfy the condition of equilibrium given by equation (3) the panel is said to be statically unbalanced. The unbalanced moment in the panel will be equal to M*, where:

$$M^* = \Sigma M_C + Q' s - b (T - H) \qquad (4)$$

To satisfy the panel equilibrium it becomes necessary to reduce the chord moments therein by an amount equal to the unbalanced value M*. This is done by imposing a displacement of the special pattern similar to that in Fig. (1-d) on that panel. The magnitude and direction of this displacement will depend on the value and sign of the unbalanced moment. Such a displacement will not affect the equilibrium of other panels, or even alter the values Q', T or H in any panel including the deformed panel itself.

Referring to the bending moment diagram produced by the special panel displacement in Fig. (1-d), the end moments may be taken as follows;

$$M_1 = n c k_1$$
$$M_2 = n (1-c) k_1$$
$$M_3 = n c k_2$$
$$M_4 = n (1 - c) k_2$$

where n is a constant

For the displacement necessary to balance the panel, the sum of the moments M_1 to M_4 must be equal to - M* and therefore:

$$n = - M^* / (K_1 + K_2) \qquad (5)$$

The additional moments thus produced in the panels will be:

$$M_1 = -\epsilon_1 M^* \quad (6\text{-a})$$

$$M_2 = -\epsilon_2 M^* \quad (6\text{-b})$$

$$M_3 = -\epsilon_3 M^* \quad (6\text{-c})$$

$$M_4 = -\epsilon_4 M^* \quad (6\text{-d})$$

where :

$$\epsilon_1 = cK_1 / (K_1 + K_2) \quad (7\text{-a})$$

$$\epsilon_2 = (1-c) K_1 / (K_1 + K_2) \quad (7\text{-b})$$

$$\epsilon_3 = cK_2 / (K_1 + K_2) \quad (7\text{-c})$$

$$\epsilon_4 = (1-c) K_2 / (K_1 + K_2) \quad (7\text{-d})$$

The terms ϵ_1 to ϵ_4 are called the balancing coefficients for the panel moments. To balance the panel the moment - M* is simply distributed on the chords according to these coefficients.

1.4 - Main Steps of Analysis

Since the open-spandrel arch is totally fixed at the springing or continuous over elastic piers, the end reactions due to any case of loading are statically indeterminate. Furthermore, it is not possible for such a system with fixed ends to produce the special type of panel displacement without producing additional moments in all panels of the system. If however we assume for the time being that we know the influence lines for the fixed end reactions, then these reactions will be easily evaluated for any case of loading. The end supports may then be removed and substituted by these reactions, and thus the open-spandrel arch is reduced to a free system acted upon by the external loads and the end reactions. In such a free system it is also possible to produce the special type of panel displacement shown in Fig. (1-d) on any panel.

Let us now consider the final bending moment diagram due to any case of loading. This diagram must satisfy the following two sets of equilibrium conditions; namely:

1. $\Sigma M = 0$ . . . . . for each joint.

2. $\Sigma M_C + Q's - b(T - H) = 0$ . . . for each panel.

Starting with all joints fixed, and referring to the open-spandrel arch in Fig. (9) with the end springings A and B free but subjected to the end reactions, let us consider the first condition of equilibrium for the joints. At this stage only the external joints C and L will be unbalanced due to the moments produced by the end reactions at A and B. These joints will then have to be balanced in the ordinary way by allowing rotation there, thus distribute the unbalanced moment on the near ends of the members framing into the joint, with half the values of the distributed moments carried over to the far ends.

Consider next the second condition of equilibrium of the panels. The

values Q' and T are known for all panels since they only depend on the external loading and end reactions. The value H so far is only due to the shear in the end verticals CM and LW resulting from the distributed and carry over moments in these verticals produced from the previous step. Finally, the sum of the chord moments ΣM_C is equal to zero for all panels except the external ones in which it is equal to the sum of the distributed and carry over moments therein due to the previous step. In other words, the value $\Sigma M_C + Q' . s - b (T-H)$ can be obtained for all panels. This will not be zero, and hence the panels are unbalanced, and the unbalanced moment in each point will be: (equation 4)

$$M^* = \Sigma M_C + Q' . S - b (T - H)$$

By producing the appropriate special type of displacement in each panel, these unbalanced moments can be distributed on the panel chords according to the balancing coefficients ϵ_1 to ϵ_4 as in article (1.3). We notice that such a balancing of any panel will not affect the equilibrium condition in any other panel.

As a result of the balancing process of the panels, the joints will become unbalanced due to the moments introduced therein, and therefore they need to be rebalanced. This again violates the equilibrium condition in the panels due to the distributed and carry over moments introduced in the chords as well as the thrust H produced by the shearing force developing in the vertical spandrel columns. The new unbalanced moments in the panels will therefore be:

$$M^* = \Sigma M_C + b H \qquad (8)$$

where M_C and H are the moments and horizontal forces resulting from the preceding step of joint rotations. These moments are distributed again according to the coefficients ϵ_1 to ϵ_4. The process thus continues, with each cycle consisting of two steps of balancing the joints then the panels, until the required degree of accuracy.

1.5 - Auxiliary Moment Charges

Due to the comparatively great slenderness of the spandrel columns compared to the arch rib and deck, especially in the external panels, the convergence of the relaxation if carried out as outlined in the previous article (1.4), will be exceedingly slow. It is then necessary that the relaxation be artificially accelerated in some fashion. This is done by exaggerating the initial values of the unbalanced panel moments by some known values.

In other words, we add extra unbalanced moments of arbitrary known values suitably chosen to the panels. These moments may be called the additional or auxiliary moment charges, using a familiar term from the electrical engineering literature. These auxiliary unbalanced moment charges are mainly introduced in the first cycle. If we denote these moments by the symbol C, then the total initial unbalanced panel moments will be given by:

$$M^* = \Sigma M_C + Q' . S - b (T - H) + C \qquad (9)$$

In the next cycles, these artificial moment charges may be retained, or they

may be increased or decreased by adding extra values with appropriate sign as may be found suitable. The total value of the unbalanced moment in any cycle will then be:

$$M^{*} = \Sigma M_C + b H + C \qquad (10)$$

in which M_C and H belong to the step of balancing the joints in the cycle and C is the new value for the auxiliary moment charge introduced in the cycle. Obviously, before the final result is obtained, all panels will have to be totally discharged of all such auxiliary moments. In other words the sum of the C values in any panel should be zero. It will be seen that when these additional moment charges are properly chosen, as illustrated by the numerical examples given hereafter, the relaxation process will converge so rapidly that the final result may be obtained after three or four cycles only.

II. Direct Evaluation of Influence Lines

2.1 - Elastic Constants for a Simple System.

Referring to Fig. (11) which represents a simple elastic member, the following are the so-called elastic constants:

a) Elastic Length $\bar{S}$
b) Position of the elastic centroid 0 which is the centroid of all elastic lengths.
c) Elastic centroidal moments of inertia I_x and I_y and product of inertia I_{xy}. These are the moments and product of inertia of all elastic lengths $d\bar{s}$ about the axes x-x and y.y through the centroid 0.

Fig. (11) also shows the magnitudes of these constants. For symmetrical members $I_{xy} = 0$. This is the case which we need to consider here.

If end B is kept fixed, the displacements of end A due to end moments or forces acting there or at the centroid 0 (which we may assume as rigidly connected to A) can be obtained in terms of these elastic constants as follows:

1. A unit horizontal force T = 1 acting at the centroid 0 will produce at both 0 and A a pure horizontal translation Δ_t equal to I_x (for symmetrical members).

$$\Delta_t = I_x$$

2. A unit vertical force V = 1 at 0 will produce a pure vertical translation $\lambda_v = I_y$
3. A unit moment M = 1 acting at A or 0 will produce a pure rotation $\phi = \bar{S}$ at 0. Hence the displacement at A which is rigidly connected to 0 will consist of:

rotation	$\phi_m = \bar{S}$
horiz. translation	$\Delta_m = -\bar{S}.\bar{Y}$
vertical translation	$\lambda_m = 0.5\ \bar{S}\ L$

On the other hand it is possible to give the elastic constants for any symmetrical member in terms of the end displacements as follows:

$$\bar{S} = \phi_m \quad (11)$$

$$\bar{Y} = - \Delta_m / \phi_m \quad (12)$$

$$I_x = \Delta_t \quad (13)$$

$$I_y = \lambda_v \quad (14)$$

2.2 - Elastic Constants for the Open+Spandrel Arch as One Elastic System

For a complex system as that of the open-spandrel arch in Fig. (9), we can assume the existence of similar elastic quantities for the whole system of arch rib, spandrel columns and deck, as given in the previous article. Such elastic constants of course will not have the same values as given in the definition for simple elastic systems as in the preceding article or in Fig. (11) but can be defined by equations (11) to (14), from the displacements of end A while end B is kept totally fixed. Thus, for the whole open-spandrel arch system we have:

S = rotation ϕ_m at A due to $M = 1$ there.
$\bar{Y}$ = - horizontal translation Δ_m at A divided by the rotation ϕ_m there, both due to an end moment $M = 1$
I_x = horizontal translation at A due to a unit horizontal force $T - 1$ at centroid 0
I_y = vertical translation at A due to a unit vertical force $V = 1$ at 0.

In fact the elastic behavior of end A due to external loads or moments acting there is exactly similar to that of a simple elastic member having the same elastic constants. Such a simple elastic member known as the "equivalent elastic system" is very useful when dealing with open-spandrel arch systems continuous over elastic piers.

2.3 - Influence Lines for the Fixed End Reactions in the Open-Spandrel Arch

The influence lines for the components T, V and M of the fixed end reaction at one support, say A, are obtained here, according to the Mueller-Breslau's principle, as the deflection lines of the loaded chord produced by imposing a unit horizontal translation, unit vertical translation or a unit rotation respectively at this support while the other support is kept fixed. The three cases of loading necessary to evaluate the elastic constants for the open-spandrel arch as given in the previous articles are:

1. Loading $M = 1$ at A. The rotation and horizontal translation of end A will be ϕ_m and Δ_m respectively, and the elastic length $\bar{S}$ and height $\bar{Y}$ of the elastic centroid are easily obtained from equations (11) and (12).
2. Loading $T = 1$ at 0 producing a pure horizontal translation $\Delta_t = I_x$ at A.
3. Loading $V = 1$ at 0 producing pure vertical translation $\lambda_v = I_y$ at A.

If now the deflection ordinates with respect to the line of springings AB due to these three cases of loading are denoted by y_m, y_t and y_v respectively, while y'_v denotes the deflection ordinates due to the third case with respect to the original unloaded position, then the influence line ordinates T, V and M

for the horizontal reaction, vertical reaction and bending moment respectively at A will be:

1. $T = y_t / I_x$ (15)

2. $V = y'_v / I_y$ (16)

3. $M = y_m / \bar{s} + \bar{Y} y_t / I_x - L y_v / 2 I_y$ (17)

Equation (17) follows from the fact that a unit rotation at A produces a similar unit rotation at the centroid 0 together with a horizontal translation equal to $\bar{Y}$ and a vertical translation equal to - L/2. Hence, the moment and forces necessary to act at 0 in order to produce these displacements will be:

$$M = 1/\bar{s}$$

$$T = \bar{Y}/I_x$$

$$V = -L/(2 I_y)$$

Since the top chord is the loaded chord, the influence line ordinates T, V and M (and hence y_m, y_t and y_v) should represent the deflections of this chord. However, if we neglect axial deformations in the spandrel columns these will be the same as the deflections in the arch rib at the joints. These latter deflections of the arch rib are obtained from the bending moments therein. The effect of axial forces in the arch rib may also be included.

2.4 - Influence Lines for Moments; Thrusts, and Shears at Internal Sections

These influence lines are also obtained as the deflection lines produced by imposing a relative unit rotation, axial translation or translation normal to the axis respectively at the required section s-s in the arch rib, spandrel columns or deck while the end springings A and B remain fixed (or rigidly attached to the rest of the structure.) It is obvious that although the two sets of internal forces and moments applied at the two faces of the cut section s-s, in order to produce the appropriate relative unit displacement there, will form a group of forces in equilibrium, yet, due to the fact that the open-spandrel arch is externally fixed, they will set up reactions at the external fixed supports. These reactions cannot be foretold by simple statics, neither can the set of internal forces applied at the cut section.

If however the end fixation is temporarily abolished, and thus reducing the system to an externally statically determinate one, hinged at one end, say B and simply supported at the other end A, the required unit relative displacement can be produced at section s-s with no end reactions set up at the supports. The set of internal forces necessary to produce this relative displacement, and the resulting B.M.D. in the open spandrel arch can now be determined as illustrated by the given numerical example hereafter. The deflection line thus produced will be called the "Free Deflection Line" since it belongs to the case with freely supported ends. The movements at the ends will consist of the rotations ϕ_1 and ϕ_2 at supports A and B respectively, and a horizontal translation Δ at A.

To re-establish the total fixations at supports A and B, it becomes necessary to remove these end movements by imposing equal and opposite

displacements therein; namely, rotations $-\phi_1$ and $-\phi_2$ at A and B respectively and translation $-\Delta$ at A. The deflection lines corresponding to these end displacements can be readily obtained from the influence lines for the end reactions by direct proportioning since these lines result from suitable unit displacements of the supports. The final deflection line obtained as the sum of all four diagrams will give the required influence line.

III. Applications

3.1 - Dimensions and Assumptions

The method is now applied to the solution of the 27 ft. span reinforced concrete open-spandrel arch models previously adopted in the experimental works of Professors Wilson and Kluge[3] and thus we can compare the experimental and mathematical results. Only the case of the open-spandrel arch with a continuous high deck (i.e. without expansion joints in the deck) is presented hereafter in detail. The cases of the low deck integral with the arch rib at the crown, with and without expansion joints in the deck, and the case of the high deck with two expansion joints are also solved and some of the results obtained are given here. The arch rib is assumed polygonal with prismatic parts between the spandrel columns. The cross section for each straight part is taken identical with that at the middle of its length. Fig. (9) shows the main dimensions of the 27' span reinforced concrete open-spandrel arch with high deck, as well as the necessary cross-sections for the deck, the spandrel columns and the arch rib. Relative values for EI in lb.ft. units numerically equal to 0.01 the moment of inertia (1/100) for the cross section in inch units are adopted. The actual values, of EI are equal to about 1,500,000 these relative values. The elastic constants and end displacements thus obtained using the relative EI values will be 1,500,000 times the real values, but the bending moment and the influence line diagrams will not be affected. Fig. (9) also shows the elastic lengths $\bar{S}$ corresponding to the relative EI values, and the stiffnesses $K = 1/\bar{S}$ for all members.

To save time and space, the distributed moments produced by the joint rotations when balancing the joints will not be computed. Instead, only the carry over moments produced at the far ends will be entered in the tabulations. These moments are half the distributed moments for prismatic bars. Thus the coefficients for distributing the unbalanced moment at each joint to the far ends of the members framing into it will be half the distribution factors, and will be called here the carry over factors. The sum of these factors at any joint will be 0.50, as shown in figure 10. In the final step in the relaxation process, the total unbalanced moment at each joint is distributed in proportion to twice these carry over factors at the joint. The values "C" for the panel centroids are next obtained from equation (1), and from these the balancing coefficients can be evaluated by equation (7). For the first external panel we have:

$$h_1 = 5.96' \quad ; \quad h_2 = 8.40' \, , \quad S = 2.50'$$

$$b = 2.44' \quad , \quad K_1 = 1.425 \, , \quad K_2 = 4.45$$

$$\therefore \quad K_1 + K_2 = 5.875$$

then:

$$C = \frac{h_2 + 2h_1}{3(h_1 + h_2)} = \frac{8.40 + 11.92}{3(5.96 + 8.40)}$$

$$= 0.472$$

$$(1-c) = 0.528$$

$$\epsilon_1 = 0.472\ (1.425) / 5.875 = 0.114$$

$$\epsilon_2 = 0.528\ (1.425) / 5.875 = 0.128$$

$$\epsilon_3 = 0.472\ (4.45) / 5.875 = 0.358$$

$$\epsilon_4 = 0.528\ (4.45) / 5.875 = 0.400$$

These values are given in Fig. (10) for all panels.

3.2 - Loading M = 1000 lb.-ft. at A

The value 1000 is taken here for convenience instead of unity. The reaction at the fixed end B will also be M = - 1000 lb.ft. The moment computations are given in table (1) for only one half of the system due to symmetry. The first cycle in the relaxation process is shown in lines (1) to (3), and it starts by balancing the external joints C and L which are the only unbalanced joints so far. For joint C the unbalanced moment M = 1000 lb. ft. is distributed and only the moments carried to the far ends M and D of the members CM and CD respectively are obtained from the carry over factors for joint C, and are entered in the table line (1). These moments are:

at D : M = 0.4865 (1000) = 487 lb.ft.

at M : M = 0.0135 (1000) = 14 lb.ft.

Before we proceed to balance the panels, we now chose the arbitrary values of the additional unbalanced moments or auxiliary moment charges C to be added to each panel. These moments are shown in brackets in line (2) and are from left to right in lb. ft. units 6000, 2400, 1600, 600 and zero. It is seen that they get smaller towards the central panel where no additional moment is necessary. Considering now equation (9) for the total initial unbalanced moments in the panels, it is found that for this case of loading. Q' = T = 0, and thus the equation reduces to:

$$M^* = \Sigma M_C + H.b + C \qquad (9\text{-a})$$

For the external panel:

$$\Sigma M_C = 3\ (487 + 0) = 1461 \text{ lb. ft.}$$

$$H = 3\ (14 + 0) / 8.44 = 5 \text{ lb.}$$

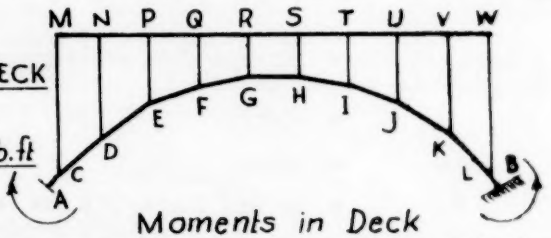
(We multiply by 3 to allow for the distributed moments which are double the

TABLE (1)

CONT. HIGH DECK

LOADING

M = 1000 lb.ft



Moments in Deck

	MC	MN	NM	ND	NP	PN	PE	PQ	QP	QF	QR	RQ	RG	RS
1	14													
2		(6000)			(2400)			(1600)			(600)			-
3		-852	-956		-371	-423		-313	-353		-141	-151		
4	36	338	385	41	165	283	39	107	165	36	32	107	14	-32
5		(-3000)			(-600)			(-500)			(-150)			-
6		-710	-795		-558	-647		-333	-376		-196	-210		-
7	16	194	154	24	121	162	30	73	121	23	25	73	11	-25
8		(-3000)			(-1800)			(-1100)			(-450)			-
9		-180	-203		-145	-166		-93	-104		-54	-58		
10	-1	12	-14	1	-1	10	0	-3	-1	-2	0	-3		0
1		—			—			—			—			
11		2	2		-9	-10		8	9		8	9		-
Balance	91	1040	1102	137	920	574	128	574	348	110	348	111	44	111
M = Σ	156	-156	-325	203	122	-217	197	20	-191	169	22	-123	69	54

Moments in Arch Rib

	CA	CM	CD	DC	DN	DE	ED	EP	EF	FE	FQ	FG	GF	GR	GH
	-1000														
1				487											
2			(6000)			(2400)			(1600)			(600)			
3			-2674	-2990		-755	-860		-442	-498		-150	-161		
4		34	1030	1301	42	361	558	37	163	251	34	37	124	12	-30
5			(-3000)			(-600)			(-500)			(-150)			
6			-2230	-2485		-1135	-1290		-470	-530		-210	-226		-
7		13	605	567	24	279	330	27	106	194	22	29	80	10	-23
8			(-3000)			(-1800)			(-1100)			(-450)			
9			-568	-635		-295	-337		-130	-146		-57	-62		
10		-1	19	-24	1	1	10	0	-11	1	0	-1	-8	0	1
-			—			—			—			—			
11			8	8		-18	-21		12	13		9	10		
12, B: 0		129	4635	3334	132	1800	1284	140	894	503	112	383	130	47	103
Σ - 1000		175	825	-437	199	238	-326	204	122	-208	168	40	-120	69	51

carry over moments in line (1) and which are not entered in the table.)

$$C = 6000 \quad \text{lb.ft.}$$

$$\therefore \; M^{*} = 1461 + 5(2.44) + 6000 = 7472 \quad \text{lb.ft.}$$

For the other internal panels $\Sigma M_C = 0$ and H = 5 lb., and the unbalanced moments in these panels from left to right will respectively be 2409, 1606, and 603 lb.ft., with no unbalanced moment in the central panel. These moments are now distributed in line (3) according to the balancing coefficients in Fig. (11).

In the first panel:

$$M_1 = - 0.114 \; (7472) = - 852 \quad \text{lb.ft.}$$

$$M_2 = - 0.128 \; (7472) = - 956 \quad \text{lb.ft.}$$

$$M_3 = - 0.358 \; (7472) = - 2674 \quad \text{lb.ft.}$$

$$M_4 = - 0.400 \; (7472) = - 2990 \quad \text{lb.ft.}$$

Due to these moments all joints become unbalanced, and so we start the second cycle (lines 4, 5 and 6) by first balancing the joints. The carry over moments produced in the far ends of the members are entered in line (4). In line (5) the new values chosen for the additional unbalanced moment charges C in the panels are given. These moments are - 3000, - 600, - 500 and - 150 lb. ft. in the four panels to the left. The negative sign opposite to that of the initial moments in line (2) means we are now starting to discharge the panels of the initial moment charges introduced in the first cycle. The total unbalanced moments are now given by equation (10); thus for the external panel:

$$\Sigma M_C = 3(338 + 385 + 1030 + 1301) = 9162 \quad \text{lb.ft.}$$

$$H = 3(36 + 34) / 8.40 = 25 \quad \text{lb.}$$

$$C = - 3000 \quad \text{lb.ft.}$$

$$\therefore \; M^{*} = 9162 + 25 \; (2.44) - 3000 = 9220 \quad \text{lb.ft.}$$

In line (6) these moments are distributed and entered in the table.

In the third cycle the panels are totally discharged of all auxiliary moment charges, and in the fourth and last cycle no additional moments are adopted. Finally the total distributed moments at the joints are entered in line (12). In joint N, for example, the sum of all moments in the three members framing into the joint (lines 1 to 11) is found to be - 2159 lb. ft. This is distributed as follows:

$$\text{in NM} \; ; \; M = 0.510 \; (2159) = 1102 \quad \text{lb.ft.}$$

$$\text{in NP} \; ; \; M = 0.427 \; (2159) = 920 \quad \text{lb.ft.}$$

$$\text{in ND} \; ; \; M = 0.063 \; (2159) = 137 \quad \text{lb.ft.}$$

(The distribution factors are twice the carry over factors shown for the joint.)

The final moments can now be obtained by summation. Considering the final moments it is interesting to notice how the external moment of 1000 lb. ft. diminishes in the arch rib towards the crown where the moment is only 50 lb. ft. or almost 5 per cent of the external moment at the springings. At the same time it is seen how the static equilibrium at any vertical section through the arch rib and deck is maintained not only by the moments therein but also by the thrusts developing in the deck. In the central panel for this case of loading, the thrust amounts to nearly 390 lb. and resists almost 90;/. of the external moment. From the bending moment diagram in the arch rib only, we can now find the deflections y_m at the spandrel columns, and also the end rotation ϕ_m and horizontal translation Δ_m at A. This is shown in table (4), in which:

$$\phi_m = 0.970 \quad \text{(radians)}$$
$$\Delta_m = -3.792 \quad \text{(ft)}.$$

(These are obviously relative values corresponding to the relative EI values used in the computations).
Hence, from equations (11) and (12):

$$\bar{S} = 0.970 \quad (\text{lb.ft.})^{-1}$$
$$\bar{Y} = 3.792/(0.970) = 3.9 \quad \text{ft.}$$

3.3 - Loading T = 1000 lb. at Centroid 0

Here also the value 1000 is adopted for convenience and the computations are shown in table (2) for only one half of the system due to symmetry. These computations can easily be followed. We notice that for this case of loading T = 1000 lbs. for the four panels to the left and Q' = 0 for all panels. Thus equation (9) for the total initial unbalanced moment in the panels becomes:

$$M^* = \sum M_C - b\,(1000 - H) + C \qquad (9\text{-b})$$

From the final B.M.D. the horizontal translation at A is found to be 5.63 (ft.) with no rotation. Thus:

$$I_x = 5.63 \text{ ft./lb.}$$

3.4 - Loading V = 100 lb. at Centroid 0

The necessary computations for this case are given in table (3). Here T = 0 and Q = 100 lb. for all panels, and thus equation (9) becomes:

$$M^* = \sum M_C + 100\,S + bH + C \qquad (9\text{-c})$$

An initial unbalanced moment equal to 3(100) = 300 lb.ft. now exists in the central panel, and this is the reason why an auxiliary moment charge C = 450 lb. ft. is here added to this panel (line 2) making a total initial unbalanced moment there equal to 750 lb. ft. From the final B.M.D., the vertical translation at A is found to be 88.10 (ft.); hence:

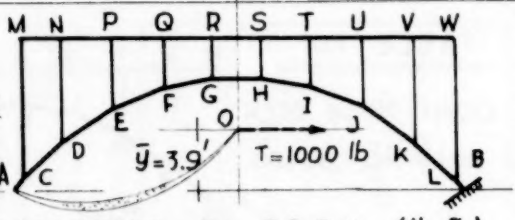
$$I_y = 88.10 \text{ ft./lb.}$$

TABLE (2)

CONT. HIGH DECK

LOADING :

T = 1000 lb.



MOMENTS IN DECK (lb. ft.)

	MC	MN	NM	ND	NP	DN	PE	PQ	QP	QF	QR	RQ	RG	RS
1	44													
2		(-4000)			(-9000)			(-7800)			(-4100)			
3		181	203		1685	1925		1750	1975		1080	1160		
4	-8	-482	-104	-71	-827	-403	-191	-660	-827	-221	-243	-660	-106	243
5		(5000)			(9000)			(7800)			(4100)			
6		337	378		471	539		730	820		510	547		
7	10	39	70	0	-3	33	1	-9	-3	1	-5	-9	3	5
8		(-1000)			—			—			—			
9		-45	-50		-17	-20		0	0		1	1		
10	2	0	-2	-2	-1	0	1	1	-1	-1	0	1	0	0
11		0	0		10	11		-3	-3		0	0		
12	1	-1	-1	-1	-2	-1	0	1	-2	0	0	1	0	0
13		-1	-1		6	7		0	0		1	1		
Balance	-6	-71	-887	-110	-744	-1670	-372	-1670	-1331	-420	-1331	-496	-195	-496
Σ =	43	-43	-394	-184	578	421	-561	140	628	-641	13	-546	-298	-248

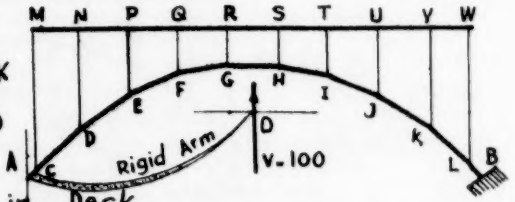
MOMENTS in ARCH RIB

CA	CM	CD	DC	DN	DE	ED	EP	EF	FE	FQ	FG	GF	GR	GH
-3300														
1			1605											
2		(-4000)			(-9000)			(-7800)			(-4100)			
3		567	635		3420	3900		2470	2785		1155	1245		
4	-9	-1788	-276	-59	-1770	-972	-183	-993	-1225	-208	-288	-756	-95	229
5		(5000)			(9000)			(7800)			(4100)			
6		1055	1180		958	1090		1030	1160		545	586		
7	6	-10	361	5	8	-6	-1	4	5	-3	8	3	-2	-7
8		(-1000)			—			—			—			
9		-140	-158		-36	-41		0	0		1	2		
10	0	-57	70	0	12	-31	0	-3	8	0	1	-2	0	-1
11		-12	-13		19	22		-4	-5		0	0		
12	0	-28	34	0	4	-15	0	-1	3	0	1	-1	0	-1
13		-4	-5		13	15		-1	-1		1	1		
B :	102	3618	-3797	-150	-2060	-3485	377	-2418	-1988	441	-1514	-555	203	441
Σ -3300	99	3201	-364	-204	568	477	-561	84	742	-652	-90	523	-302	-221

TABLE (3)

CONT. HIGH DECK

LOADING V=100



a. Moments in Deck

	MC	MN	NM	ND	NP	PN	PE	PQ	QP	QF	QR	RQ	RG	RS
1	-17													
2		(-8000)			(-2400)			(-700)			(+700)			(450)
3		1100	1235		327	370		79	90		-232	-249		-203
4	-47	-398	-500	-49	-101	-333	-26	31	-101	7	95	31	37	95
5		(8000)			(2400)			(700)			(-700)			(-450
6		400	450		274	313		11	13		0	0		-164
7	2	-19	20	0	1	-16	0	-3	1	2	0	-3	0	0
-		-			-			-			-			-
8		-4	-4		5	7		-5	-5		-3	-3		0
Balan	-82	-935	-846	-104	-708	-192	-43	-193	57	18	58	192	75	192
Σ	-164	144	-355	-153	-202	149	-69	-80	55	27	-82	-32	112	-80

b. Moments in Arch Rib

	CA	CM	CD	DC	DN	DE	ED	EP	EF	FE	FQ	FG	GF	GR	GH
	1300														
1				-632											
2			(-8000)			(-2400)			(-700)			(700)			(450
3			3460	3865		660	755		112	127		-249	-268		-173
4		-44	-1230	-1683	-49	-240	-667	-22	+31	-167	10	-102	23	37	81
5			(8000)			(2400)			(700)			(-700)			(-450
6			1260	1410		555	635		16	18		0	0		-140
7		2	2	7	-2	2	1	0	9	1	-1	0	7	0	0
-			-			-			-			-			-
8			-12	-13		11	13		-7	-7		-3	-3		0
Bal.		-128	-4610	-2464	-97	-1330	-487	-52	-337	85	19	65	201	75	160
Σ=	1300	-170	-1130	490	-148	-342	250	-74	-176	57	28	-85	-40	112	-72

TABLE (4)

OPEN-SPANDREL ARCH

Cont. High Deck

LOADING		DEFLECTION y AT D	E	F	G	H	I	J	K	OTHER RESULTS & INFORMATIONS
Moment M = 1.0 lb.ft (Table 1)	y_m	-1.165	-1.923	-2.358	-2.543	-2.543	-2.358	-1.923	-1.165	Elastic Length $\bar{S}$ = 0.97 Height $\bar{Y}$ = 3.90'
T = 1.0 lb. at O (Table 2)	y_t	0.950	2.700	4.165	4.940	4.940	4.165	2.700	0.950	I_x = 5.63
V = 1.0 lb. at O (Table 3)	y'_v	84.30	65.55	63.85	50.78	37.32	24.25	12.55	3.80	y' relative to fixed end. I_y = 88.10
(y relative to AB)	y_v	6.0	7.03	5.12	1.84	-1.84	-5.12	-7.03	-6.0	

INFLUENCE LINES FOR M, T & V at END "A"

1. Horiz. Reaction $T = y_t/I_x$		0.174	0.480	0.740	0.880	0.880	0.740	0.480	0.174	I_x = 5.63
Experimental Values.		0.164	0.464	0.764	0.896	0.896	0.764	0.464	0.164	
2. Vertical Reac. $V = y'_v/I_y$		0.957	0.858	0.725	0.575	0.425	0.275	0.142	0.043	I_y = 88.10
Experimental Values		0.954	0.862	0.728	0.578	0.422	0.272	0.138	0.046	
(3)	$y_m/\bar{S}$	-1.20	-1.99	-2.42	-2.62	-2.62	-2.42	-1.99	-1.20	= 1.03 y_m
Fixed end moment	$\bar{Y}y_t/I_x$	0.68	1.87	2.90	3.43	3.43	2.90	1.87	0.68	= 0.693 y_t
M at A	$-Ly_v/2I_y$	-0.92	-1.08	-0.78	-0.28	0.28	0.78	1.08	0.92	= - 0.153 y_v
$M = y_m/\bar{S} + \bar{Y}y_t/I_x - Ly_v/2I_y$		-1.44	-1.20	-0.30	0.53	1.09	1.26	0.96	0.40	= Σ
Experimental values		-1.42	-1.28	-0.30	0.55	1.17	1.37	0.99	0.34	

3.5 - I. Lines for the Fixed End Reactions

The deflection ordinates y_m and y_t for the loadings M = 1 and T = 1 and the deflections y_v and y'_v for the loading V = 1 are obtained from the final B.M. Diagrams and are given in table (4). From these the influence line ordinates are computed from equations (15) to (17), and are presented in table (4) together with the experimental values. The agreement between both results is remarkable. Fig. (3) also shows these influence lines as well as the influence lines for the arch rib only to indicate the extent of the deck participation.

3.6 - B.M.D. For Any Particular Loading: (Figure 4)

As an example, the case of a single concentrated load P = 1000 lbs. acting at joint P is now considered. From the influence lines Fig. (3) or table (4), it is found that the end reactions are:

At A :	M =	1200 lb.ft.	clockwise
	T =	480 lb.	from left to right.
	V =	858 lb.	upwards.
At B :	M =	-960 lb.ft.	anticlockwise
	T =	-480 lb.	from right to left.
	V =	142 lb.	upwards.

The initial unbalanced moments in the panels are obtained from equation (4) are first evaluated. For the four panels to the right H = - 3 lbs., and T = - 480 lbs. while in the panels to the left H = 5 lbs. and T = 480 lbs. Also Q' = 850 lbs. for the first two panels to the left of P and Q = - 142 lbs. in the other seven panels to the right. For the panels to the left H is so far equal to the shear in the first column CM, and in the panels to the right it is equal to the shear in LW. The initial unbalanced moments thus obtained are shown in figure (4). The rest of the computations given there follows in the usual way and needs no further explanation. It is noticed that during the relaxation process the sum of the shearing forces in the spandrel columns is not zero, but in the final result it should become zero (or very nearly so). It is also interesting to point out that only three cycles were sufficient to get the final B.M.D. which is shown in figure (4).

3.7 - Influence Line for the Thrust at the Crown of the Arch Rib (Table 5)

As previously explained in article (2.4), the system is first reduced to a simply supported one, in which a unit relative axial translation is to be produced at the crown of the arch rib in the central panel. The most convenient way for producing such a relative translation is shown in table (5). With the bottom chord GH supposed to be temporarily disconnected from the rest of the system at G and H, two equal and opposite rotations $\phi = 1/2h$ and $\phi = -1/2h$ are imposed at R and S respectively, where "h" is the height of the central panel = 2.25'. The same angles of rotation are also imposed on the ends G and H respectively of the bottom chord GH. It is seen that the four panels to the left will thus rotate as a rigid body through the angle ϕ about R while the four panels to the right will rotate through an angle $-\phi$ about S. Such rotations are possible in a simply supported system as it is now the case. No bending moments are therefore produced except in the chords of the central panel. Considering the deformed shape of the open-spandrel arch as

Figure (3)

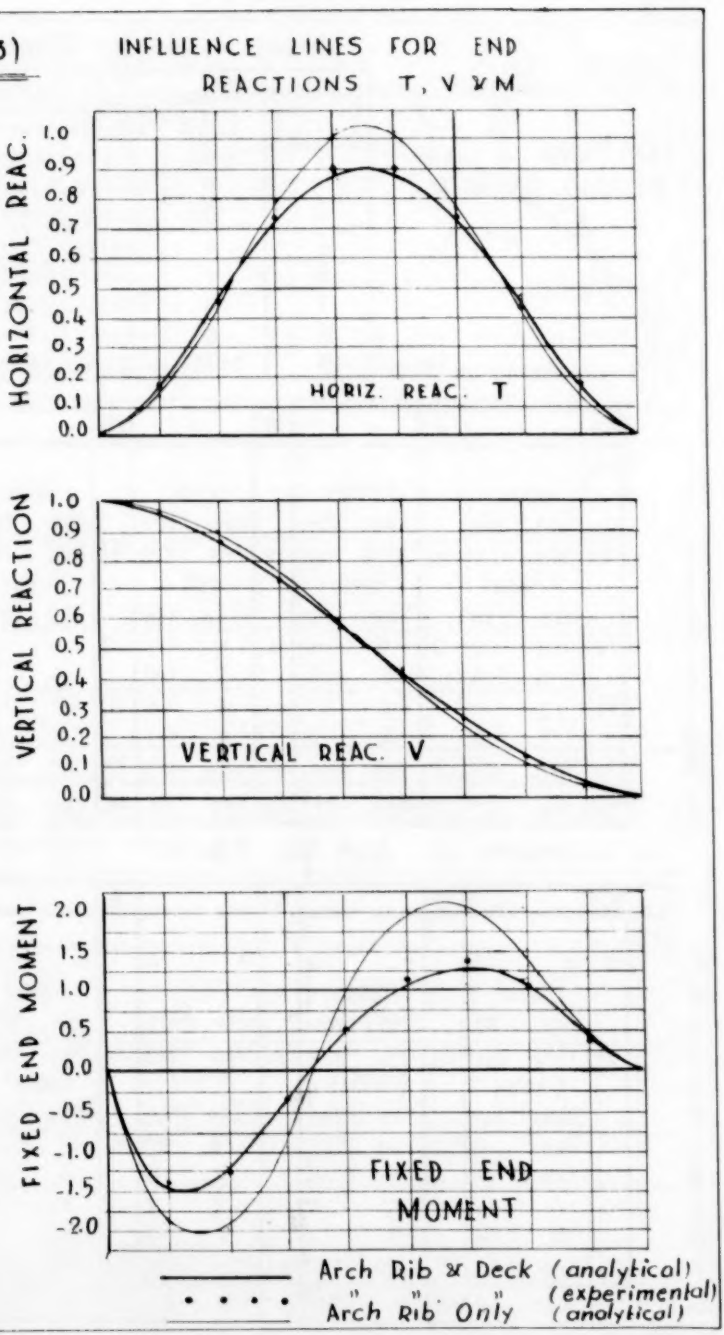

INFLUENCE LINES FOR END REACTIONS T, V & M
HORIZONTAL REAC.
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
HORIZ. REAC. T
VERTICAL REACTION
VERTICAL REAC. V
FIXED END MOMENT
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
FIXED END MOMENT
Arch Rib & Deck (analytical)
" " " (experimental)
Arch Rib Only (analytical)

FIGURE (4)

LOADING

P = 1000 lbs at P

Initial Unbalanced Moments.

Panel nº	ΣM_c	+ s Q	-(T-H) b	$M^* - \Sigma$
1	-3 x 515	2.5 x 858	-(485) 2.44	-582
2	0	3 x 858	-(485) 1.96	1623
3	0	-3 x 142	-(485) 1.2	-1008
4	0	-3 x 142	-(485) 0.55	-693
5	0	-3 x 142	0	-426
6	0	-3 x 142	(483) 0.55	-164
7	0	-3 x 142	(483) 1.2	144
8	0	-3 x 142	(483) 1.96	508
9	-3 x 361	-2.5 x 142	(483) 2.44	-277

a. Moments in Deck

	M			N			P			Q			R	
1	-14													
2		(5000)			(6500)			(00)			(-600)			(-1600)
3		-505	-565		-1250	-1430		196	222		302	323		545
4	21	462	239	60	278	387	78	-113	278	-36	-181	-113	-69	-80
5		(-5000)			(-6500)			(0)			(600)			(1600)
6		-456	-512		-45	-52		-340	-385		273	294		-56
7	-2	-5	-13	2	9	-4	-4	11	9	-2	5	11	-2	6
8		8	9		4	5		9	10		-4	-4		-1
9	-1	-3	0	1	-4	-2	-1	-4	-4	1	-2	-4	1	-1
10		7	8		4	5		6	7		0	0		0
B	40	448	907	112	756	564	125	564	-213	-67	-213	-355	-140	-355
Σ=	44	-44	73	175	-248	-527	198	329	-76	-104	180	152	-210	58

b. Moments in Arch Rib (lb. ft.)

		C			D			E			F			G	
	1059														
1		0	0	-515		0	0		0	0		0	0		
2			(5000)			(6500)			–			(-600)			(-1600)
3			-1580	-1765		-2550	-2900		277	312		322	347		465
4		21	1520	769	57	725	827	61	-160	505	-36	-188	-122	-71	-54
5			(-5000)			(-6500)			–			(600)			(1600)
6			-1435	-1605		-93	-105		-480	-540		293	315		-47
7		-1	46	-51	-1	-40	25	2	-8	-27	3	-5	-6	2	-5
8			26	29		9	11		12	14		-4	-4		-1
9		0	17	-34	0	-12	9	-1	5	-8	-1	3	4	-1	1
10			21	24		9	10		9	10		0	0		0
B		10	396	3188	126	1730	1333	145	928	-329	-73	-251	-381	-140	-303
Σ=	1059	30	-1089	40	182	-222	-790	207	583	-63	-107	170	153	-209	56

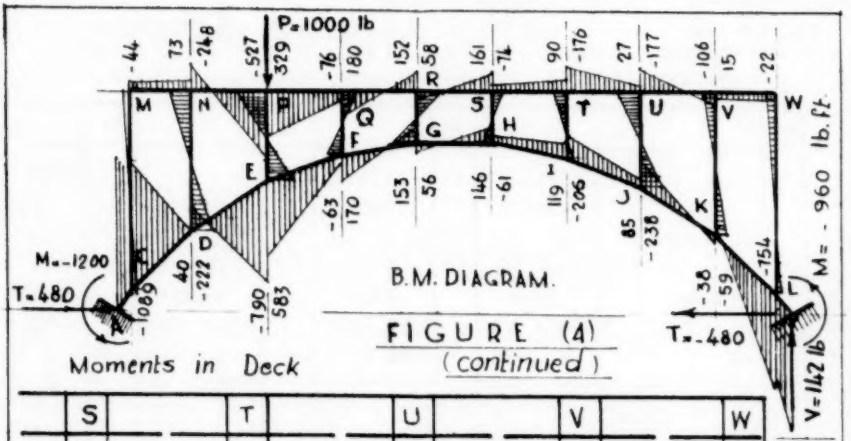


FIGURE (4)
(continued)

Moments in Deck

	S			T			U			V			W	
													-10	1
(-1600)		(800)			(1000)			(3500)			(2500)			2
545		-159	-148		-252	-223		-705	-616		-286	-254		3
-181	-25	86	-80	29	210	86	52	193	210	32	121	230	11	4
(1600)		(-800)			(-1000)			(-3500)			(-2500)			5
-56		148	138		-312	-276		-93	-82		-272	-242		6
5	-3	3	6	0	8	3	-1	-2	8	1	0	-2	-1	7
-1		-1	0		-3	-3		-9	-8		-5	-4		8
-2	0	-2	-1	1	3	-2	0	1	3	1	3	1	0	9
0		0	0		-5	-4		-8	-7		-3	-2		10
-149	-59	-149	175	56	175	446	99	446	386	57	457	251	22	B.
161	-87	-74	90	86	-176	27	150	-177	-106	91	15	-22	22	Σ

Moments in Arch Rib.

	H			I			J			K			L		
														743	
		0	0		0	0		0	0		-361	0	0		1
(-1600)		(800)			(1000)			(3500)			(2500)				2
465		-170	-158		-354	-314		-1430	-1256		-893	-800			3
-150	-32	99	-68	27	336	129	46	430	485	29	390	792	11		4
(1600)		(-800)			(-1000)			(-3500)			(-2500)				5
-47		159	148		-440	-390		-190	-165		-848	-760			6
-4	3	-1	-7	1	-5	-1	2	19	-7	0	-20	34	0		7
-1		-1	-1		-5	-4		-19	-17		-14	-13			8
3	0	3	1	-1	1	4	1	10	1	0	-14	18	0		9
0		0	0		-7	-6		-16	-14		-9	-9			10
-120	-56	-150	204	60	268	667	104	958	935	68	1710	-16	0	0	B
146	-85	-61	119	87	-206	85	153	-238	-38	97	-59	-754	11	743	Σ

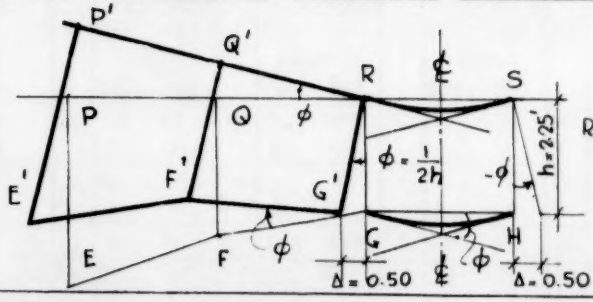
TABLE (5) I.L. for THRUST AT CROWN

a. Moments in Deck

	M			N			P			Q			R	
1														5290
2		-1000			-2000			-2800			-9200			-9500
3		114	128		308	352		546	616		2140	2300		2560
4	-5	-111	-52	-13	-202	-93	-45	-595	-202	-177	-2120	-595	-780	2120
5		1000			2000			2800			9200			9500
6		112	126		104	119		573	647		1720	1840		-2560
7	0	10	2	0	9	8	-1	28	9	-9	-5	28	-2	5
8		0	0		-3	-3		6	7		8	9		0
9	0	-2	-5	0	-8	-2	1	-2	-8	-1	-8	-2	1	8
10		0	0		-2	-3		4	5		8	9		0
B.	-10	-108	-200	-25	-167	-402	-89	-402	-1136	-358	-1136	-4275	-1681	-4275
Σ	-15	15	-1	-38	39	-24	-134	158	-62	-545	607	-686	-2462	3148

b. Moments in Arch Rib

	C			D			E			F			G	
1														4490
2		(-1000)			(-2000)			(-2800)			(-9200)			-9500
3		358	400		616	714		770	868		2290	2470		2190
4	-5	-320	-174	-14	-412	-174	-45	-796	-286	-187	-2115	-606	-835	1680
5		1000			2000			2800			9200			9500
6		350	392		211	242		810	910		1840	1970		-2190
7	0	-1	-12	1	10	-1	2	-41	-7	9	-4	-31	-2	4
8		1	1		-6	-6		9	10		9	10		0
9	0	8	0	0	10	5	-2	-4	7	-1	4	-3	-3	-4
10		-1	-1		-5	-5		6	6		9	10		0
B.	-10	-380	-630	-25	-342	-823	-89	-572	-1696	-376	-1290	-4230	-1560	-3360
Σ	-15	15	-24	-38	62	-48	-134	182	-188	-555	743	-410	-2400	2810



Imposed Relative Translation (Free System)

shown in figure (5) it is clear that the only relative displacements therein is a horizontal translation equal to 0.50 at the two ends G and H of the bottom chord GH of the central panel. The moments necessary to produce these end rotations ϕ and $-\phi$ at the left and right ends respectively of each chord is equal to $\pm 2\phi K = \pm K/h$ where K = stiffness of that chord. Thus; in the top chord:

$$M = \pm 1.19/2.25 = + 0.525 \text{ (lb.ft./ft.)}$$

and in the bottom chord.

$$M = + 1.01/2.25 = + 0.449 \text{ (lb.ft./ft.)}$$

These are the initial moments entered in line (1) table (5). All moments in this table are multiplied by 10,000 for convenience.

We shall now assume that the continuity is re-established between the bottom chord GH and the rest of the open-spandrel arch system. Considering the moments in line (1), it is seen that no unbalanced moments so far exist in any panels since $Q' = T = H = \Sigma M_C = 0$ in all panels. Thus the total initial unbalanced moments will only be the auxiliary moment charges "C" entered in the first cycle Line (2). The rest of the computations can be easily followed.

From the final moments, the end displacements are found to be:

$$\text{at A} : \phi_1 = 0.198 \text{ (radians) and } \Delta = -3.6 \text{ (ft.)}$$

$$\text{at B} : \phi_2 = -0.198 \text{ (radians).}$$

In the next step, these displacements need to be removed. Thus the necessary displacements at A and B required in order to bring these ends back to the original fixed position are:

at A:

$$\text{rotation} = -\phi_1 = -0.198 \text{ (radians)}$$

$$\text{translation} = -\Delta = 3.60 \text{ (ft.)}$$

at B:

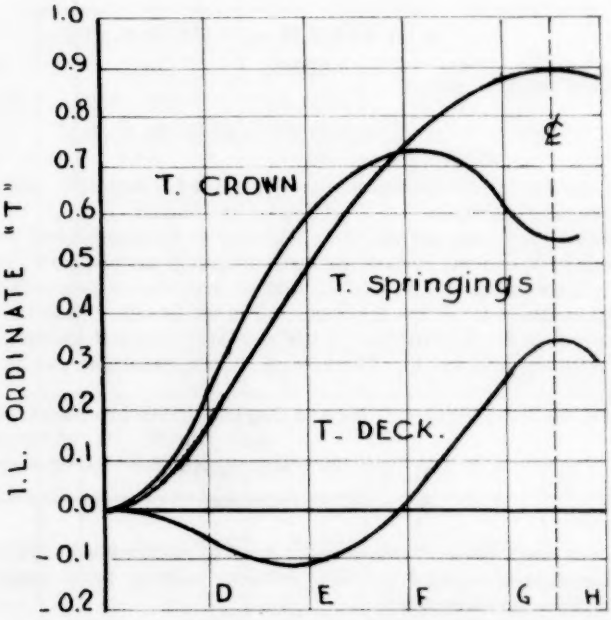
$$\text{rotation} = -\phi_2 = +0.198 \text{ (radians)}$$

In figure (5), the deflection ordinates for all these four cases are tabulated and added to get the final influence line ordinates. By subtracting these values from those for the end thrust T at the springings (Fig. 3 and table 4) we can obtain the influence line of the thrust in the deck at the central panel. These influence lines are shown in figure (5). It is interesting to notice that tensile forces develop in the deck by vertical loads acting on the external three panels on both sides. For such loading, the thrust in the arch rib at the crown is greater than the horizontal reaction at the springings.

3.8 - Effect of Expansion Joints in Deck

Assume now that two expansion joints are introduced in the deck of the previous open-spandrel arch model in the third and seventh panels which will

FIGURE (5)



I. L. COMPUTATION

1_ THRUST in ARCH RIB at CROWN

IMPOSED DISPLACEMENT	DEFLECTION AT D	E	F	G	H
(1) Free Deflec. Line	-0.594	-1.183	-1.752	-2.231	-2.231
(2) $-\phi_1$ at A = -0.198	0.285	0.238	0.061	-0.105	-0.216
(3) $-\phi_2$ at B = 0.198	-0.079	-0.190	-0.249	-0.216	-0.105
(4) $-\Delta$ at A = 3.60	0.625	1.728	2.668	3.168	3.168
T at Crown = Σ	0.237	0.593	0.728	0.616	0.616

2_ THRUST in DECK at CROWN.

Subtract from TA at springings	-0.063	-0.113	0.012	0.264	0.264

now reduce to the arch rib there only. The balancing coefficients for the remaining seven panels will be unchanged. Only the carry-over factors for moments at joints E, F, P, Q, I, J, T and U need to be modified. Figure (6) shows the bending moment diagrams due to the three main cases of loading M = 1000 (lb. ft.), T = 1000 (lb.) and V = 100 (lb.) together with the influence line ordinates for the reaction components at end A. As an example, the moment computations for the case M = 1000 (lb. ft.) only is given in table (5). Comparing these diagrams with those obtained for the case of the continuous deck it is seen that the introduction of the expansion joints has greatly reduced the deck participation.

3.9 - Case of Low Deck

To see the effect of the height of the deck on the distribution of stresses on the open-spandrel arch system, the case of a low deck integral with the arch rib at the crown is also considered. The main dimensions and cross sections are kept the same as for the case of the high deck shown in figure (9) except for the heights of the spandrel columns which are now reduced by 1' 8". The system now consists of four panels on each side of the central part which, to simplify the analysis, is considered as a prismatic member, with the two spandrel columns on both sides as infinitely rigid. It is obvious that in this case no moments propagate across the central part from one side to another. Figure (7) shows the bending moment diagrams for this system due to the three main cases of loading and the influence line ordinates for the fixed end reactions. To illustrate the extent of deck participation in all three constructions of the open-spandrel arch system, the bending moment diagrams for them all as well as for the case of arch rib only, due to the single concentrated load P = 1000 lbs., are grouped together in figure (8).

IV. Deck Participation

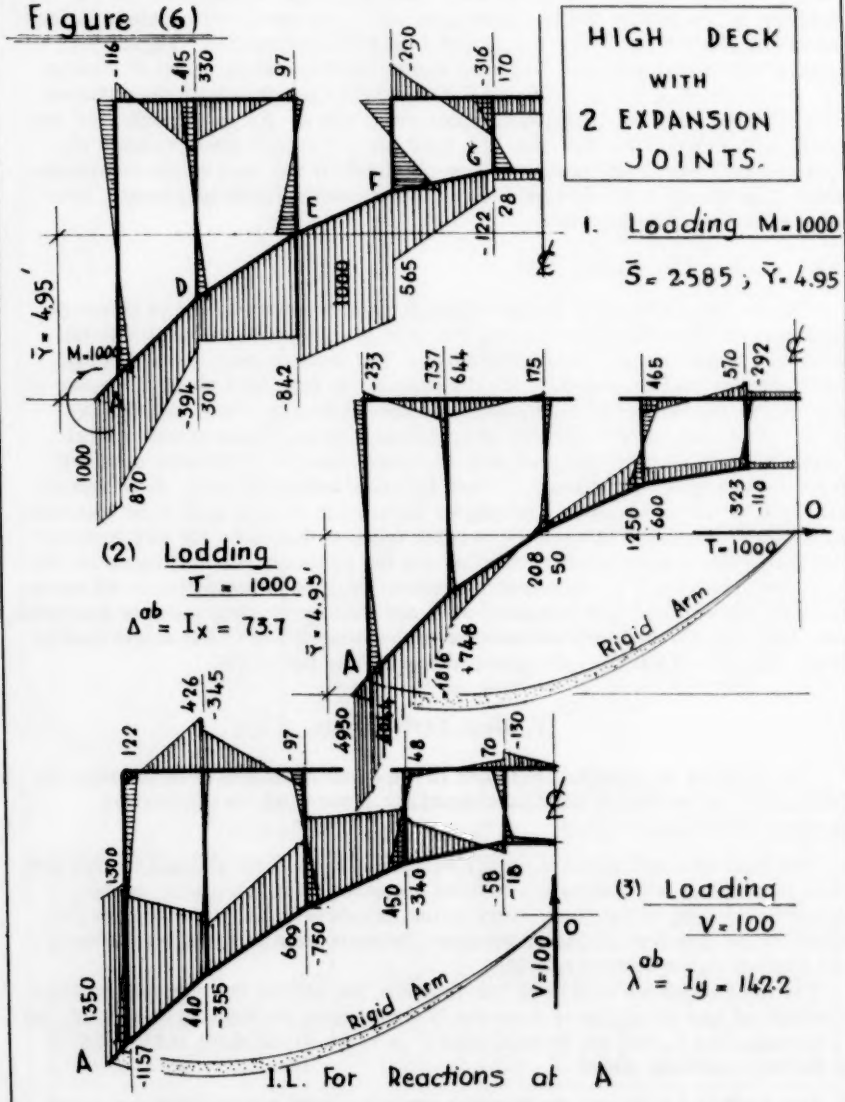
The study of the results presented in the given examples here justifies the following conclusions on deck participation. More work should still be favored in this field.

1. The total external moment on any vertical section through the arch rib and deck is resisted not only by the bending moments therein but also by the thrust developing in the deck. This thrust, which is due to the moments produced in the spandrel columns, becomes the main item in resisting the external moment in the central panels.

For the continuous high deck construction, the part of the external moment resisted by this thrust, in or near the central panel, for the end loading M, the centroidal load T, and the vertical load P is respectively 0.90, 0.85 and 0.77 of the total moment there.

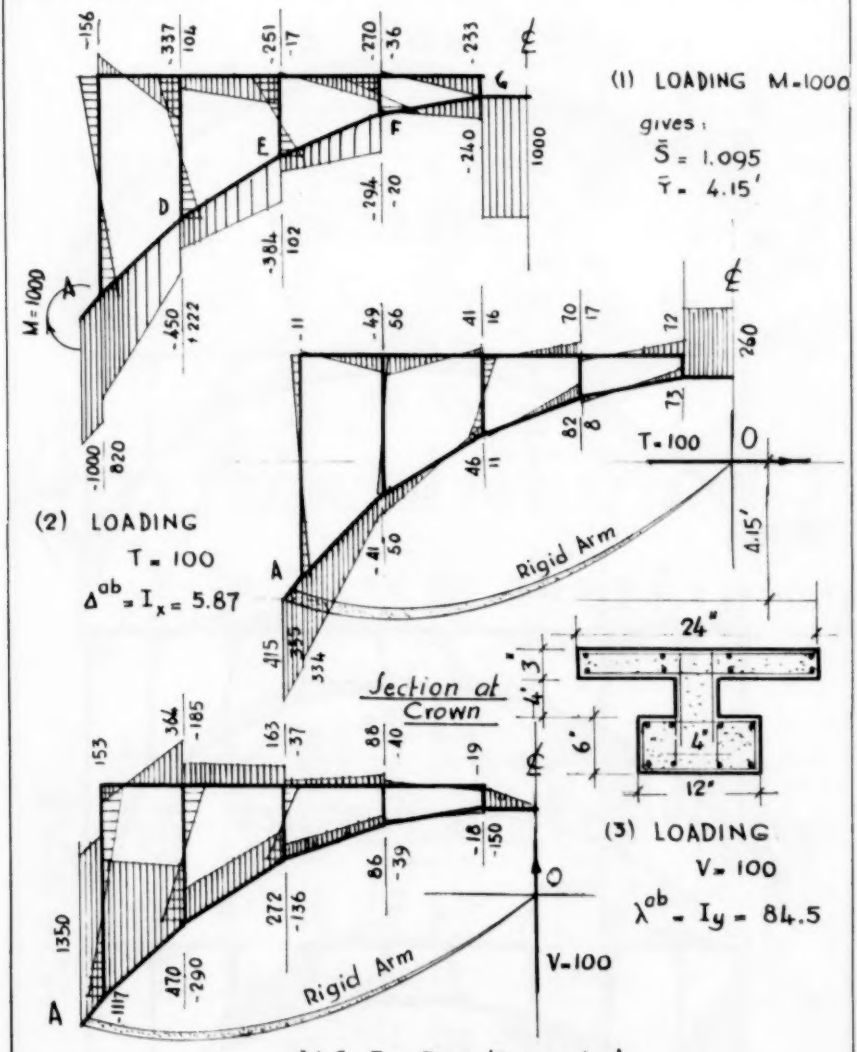
2. The fixed end moments for the open-spandrel arch construction are much less than those for the same arch rib only, especially when no expansion joints are introduced in the deck. It therefore follows that the moments developing in the arch rib will be much reduced due to the deck participation. At the same time, moments will develop in the deck, of almost the same pattern, and of magnitudes depending on the relative stiffnesses of the arch rib and deck. The bending moment diagrams in figure (8) show to what extent this is true.

Figure (6)



Ordinate at	D	E	F	G	H	I	J	K
Horiz. Reaction T	0.17	.488	.761	.850	.850	.761	.488	.170
Vertical Reaction V	.973	.906	.776	.595	.405	.224	.094	.027
Moment M	-1.69	-1.88	-.93	0.32	1.36	2.01	1.58	0.59

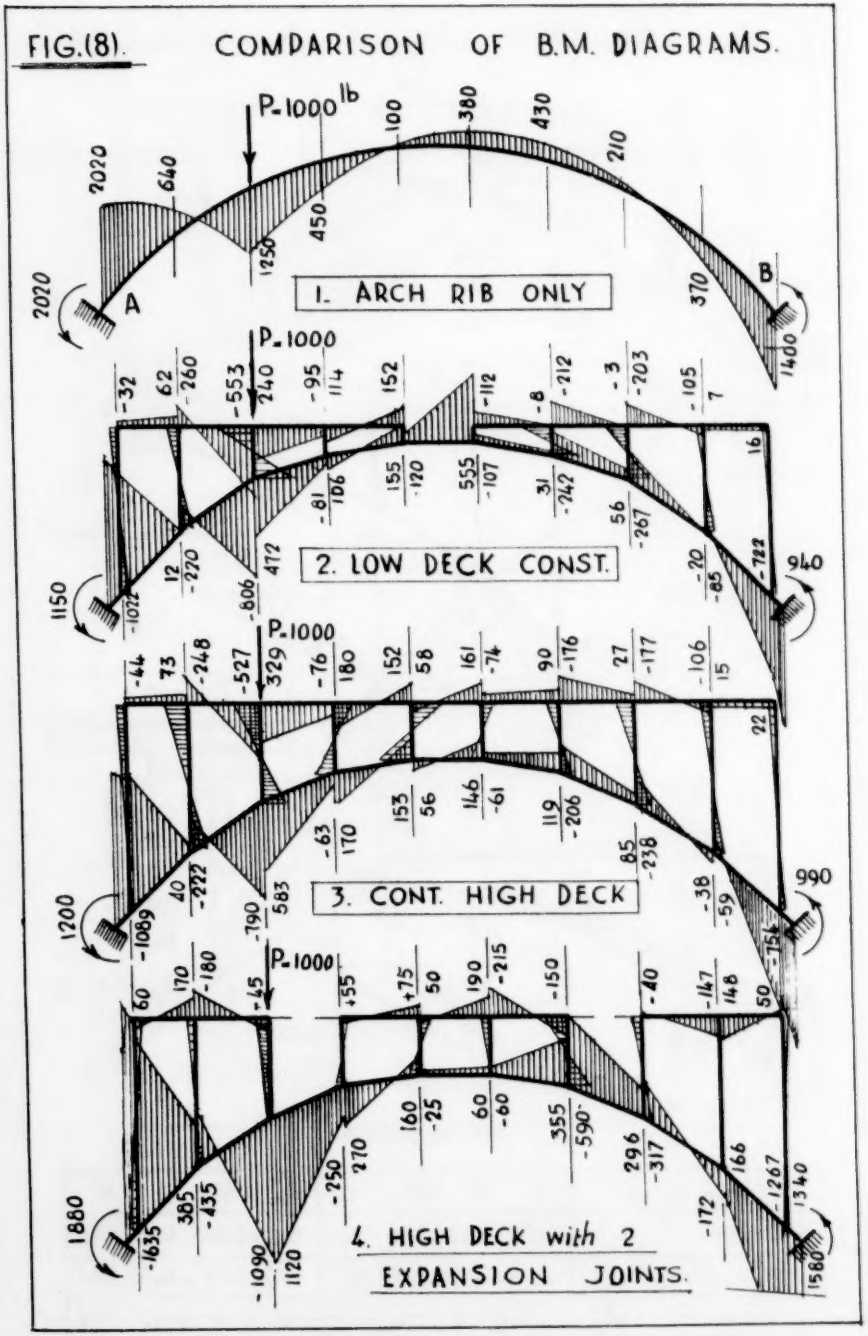
FIG. (7) - LOW DECK (Integral with arch rib.)



I.L.s For Reactions at A

Ordinate at	D	E	F	G	H	I	J	K
THRUST T	0.176	.478	.735	.875	.875	.735	.478	.176
VERT. REAC. V	0.956	.855	.722	.575	.425	.278	.145	.044
MOMENT M	-1.47	-1.15	-.34	0.45	1.01	1.18	0.94	.335

FIG.(8). COMPARISON OF B.M. DIAGRAMS.



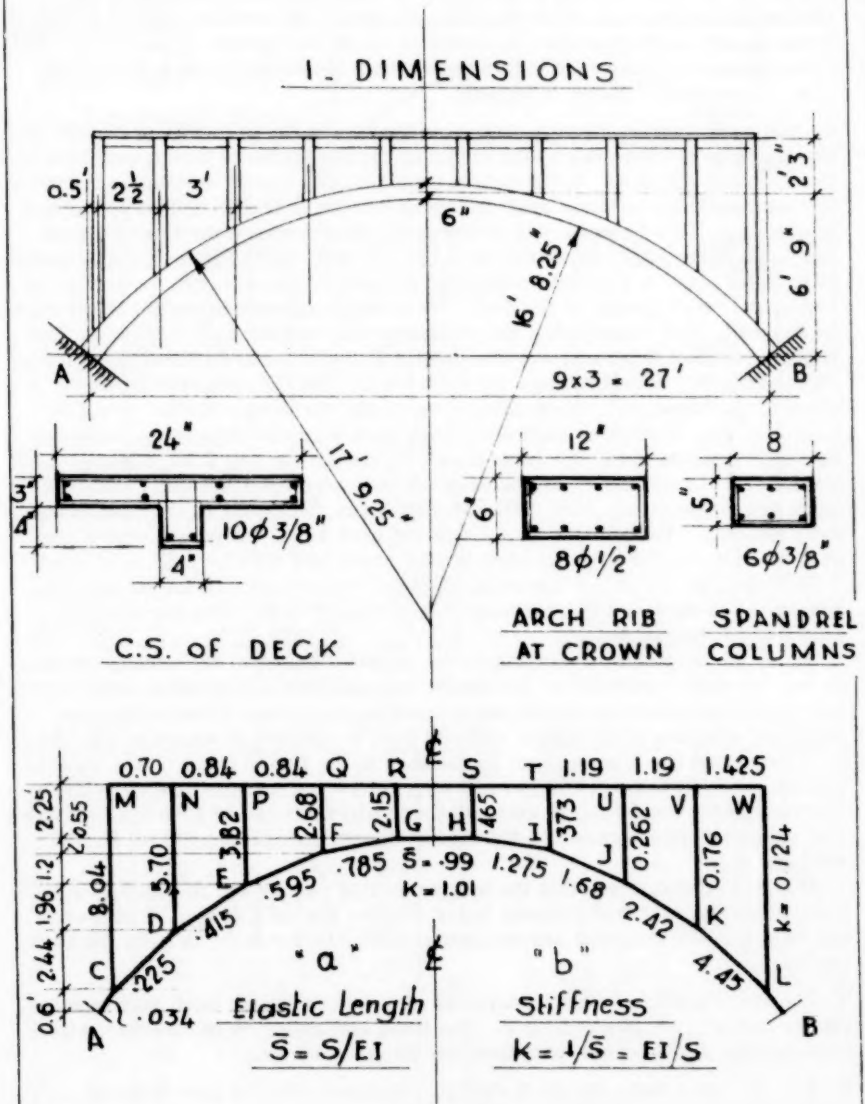


FIGURE (9)

R.C. OPEN-SPANDREL ARCH.

3. The height of the spandrel columns, when all other dimensions are kept the same, is found to have a very small influence on the moments at the springings or in the arch rib or deck. However, the moments in the spandrel columns are much increased by lowering the deck. In fact, it is doubtful that these slender columns can safely resist such big moments as may develop from some special cases of loading.

4. The most serious item in deck participation is the presence of thrusts in the arch rib and deck due to the shearing forces produced by the moments in the spandrel columns. In the panels where tensile forces develop in the deck, the horizontal force in the arch rib there will be in excess of that at the end springings. This is usually expected in the panels where the line of thrust lies outside the panel below the arch rib. In such cases the deck participation is harmful since it produces additional thrusts in the arch rib, as well as unexpected tensile forces in the deck. An example of such harmful contribution is met with when considering the shrinkage and temperature stresses. The bending moment diagrams for this case are proportional to those given for the loading "T" acting through the centroid 0. For the same temperature change, the horizontal thrust developing at the springings for the cases of arch rib only, continuous high deck, high deck with two expansion joints and low deck constructions are respectively in the ratios 1-0, 2.30, 1.70 and 2.20. Tensile forces developing in the deck for the three open-spandrel constructions are respectively 2.50, 1.05 and 4.50 times the thrust in the case of the arch rib only. The horizontal force in the arch rib in the panels where the deck is subjected to the maximum tensile force will then be 4.80, 2.75 and 6.70 (times the thrust for the arch rib only) respectively. Both the tensile forces in the deck and the excessive compression in the arch rib in these cases are remarkable.

Since the moments developing in the spandrel columns are usually greater in the low deck construction, the thrust thus produced in the deck and arch rib will rapidly increase in magnitude by lowering the deck. Considering, for example, the case of the single vertical load P - 1000 lbs. shown in Fig. (8), it is found that the tensile force in the deck of the fourth panel on the right is 610 lbs. and 175 lbs. for the low and high deck constructions respectively. The horizontal forces in the arch rib there will increase to 1090 lbs. and 655 lbs. respectively compared to 480 lbs. at the springings and 465 lb. for the arch rib only.

It is also obvious now that the approximation used often, in which it is assumed that the spandrel columns being slender are as if hinged at both ends and thus will not transmit any horizontal forces to the deck, is quite misleading.

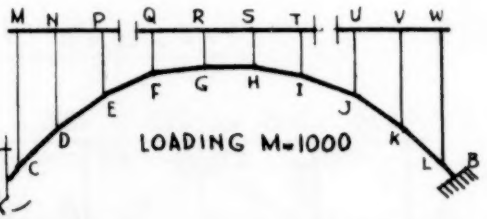
5. The introduction of expansion joints will not abolish the deck participation totally, but will greatly reduce it. The fixed end moments in this case will generally be slightly less than those for the arch rib only.

6. For the cases when the arch rib axis coincides with the line of thrust, which may be achieved for dead loads, there will be no chance for the deck participation to take place as long as rib shortening is disregarded.

It is mainly, then, under live loads, temperature change, shrinkage, and displacement of supports that the superstructure becomes active and deck participation must be considered.

TABLE (6)

HIGH DECK WITH
2 Expansion Joints



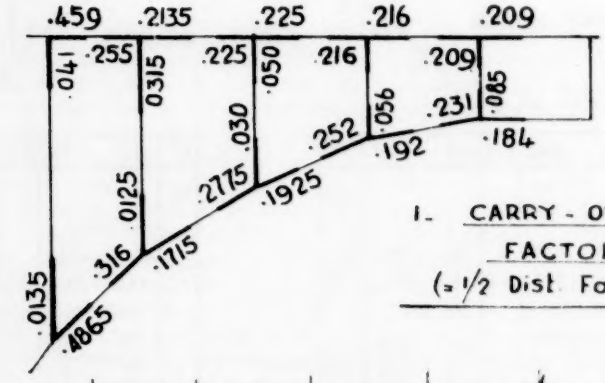
a. Moments in Deck

	MC	MN	NM	ND	NP	PN	PE	QF	QR	RQ	RG	RS
1	14						-49	113				
2		(5800)			(300)				(1400)			-
3		-830	-930		161	184			-612	-658		0
4	35	196	375	32	-55	164	-18	73	138	190	27	-138
5		-5800			(-300)				(-1400)			-
6		-249	-279		-143	-164			-152	-162		0
7	-1	18	8	0	7	15	-5	3	17	-23	-7	-17
8		-4	-4		6	7			9	10		0
9	0	-4	-6	1	-7	-4	-1	2	8	-11	-4	-8
10		-4	-4		4	4			5	5		0
Bal.	68	761	425	52	357	-109	-24	99	315	333	130	333
Σ	116	-116	-415	85	330	97	-79	290	-290	-316	146	170

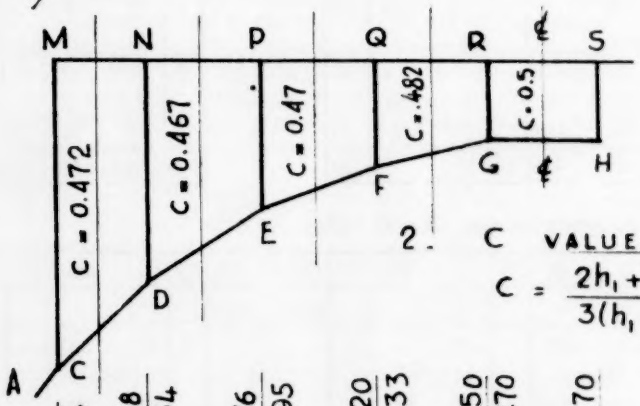
b. Moments in Arch Rib

	CA	CM	CD	DC	DN	DE	ED	EP	EF	FE	FQ	FG	GF	GR	GH
	-1000								1000	-1000					
1		0	0	487		-451		0			0	0	387		0
2			(5800)			(300)						(1400)			-
3			-2602	-2908		326	373					-655	-705		0
4		33	805	1265	24	-168	436	-12			60	74	255	54	-58
5			(-5800)			(-300)						(-1400)			
6			-782	-870		-291	-332					162	-174		0
7		1	12	-27	2	-42	7	1			-7	-18	11	7	14
8			-12	-13		12	14					10	11		0
9		0	22	0		-9	12	-1			-4	-10	6	3	8
10			-13	-14		7	8					5	6		0
B. 0		96	3440	1686	67	916	-1460	-146	0	0	386	1321	81	80	64
Σ	-1000	130	870	-394	93	301	-842	-158	1000	-1000	435	565	-122	94	28

FIGURE (10)

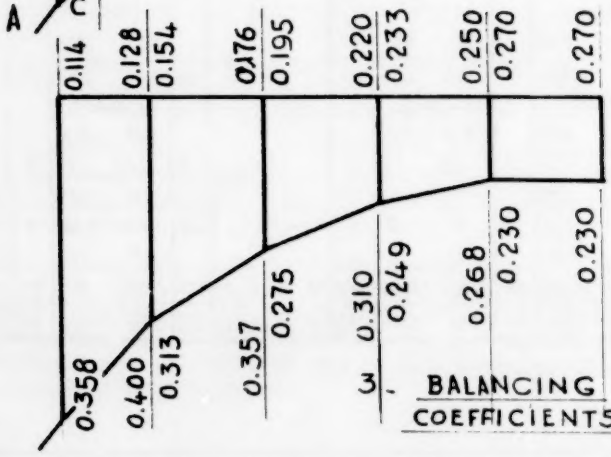


1. CARRY - OVER FACTORS (= 1/2 Dist. Factors)



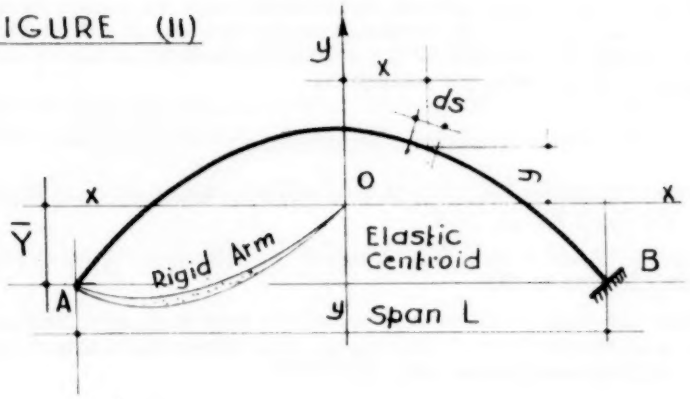
2. C VALUES

$$C = \frac{2h_1 + h_2}{3(h_1 + h_2)}$$



3. BALANCING COEFFICIENTS.

FIGURE (II)



ELASTIC CONSTANTS
For
SIMPLE ELASTIC SYSTEMS

Elastic Length $= \bar{S} = \int ds/EI$
$= \int d\bar{s}$

Elastic Centroid : O , height $\bar{Y}$
= centroid of lengthes $d\bar{s}$.

Elastic Moments of Inertia about Centroidal Axes :

$$I_x = \int y^2 \, d\bar{s}$$

$$I_y = \int x^2 \, d\bar{s}$$

$$I_{xy} = \int xy \, d\bar{s}$$

For symmetrical systems $I_{xy} = 0$.

ACKNOWLEDGMENTS

The writer is greatly indebted, and wishes to express sincere thanks and appreciation, to Dr. Kurt H. Gerstle, Assistant Professor of Civil Engineering, University of Colorado, for the valuable assistance and advice in the preparation of this paper for publication.

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THE DESIGN OF THE MAIN TOWERS OF THE MACKINAC BRIDGE

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(Proc. Paper 1565)

SYNOPSIS

The various aspects of the design of the main towers of the Mackinac Straits Bridge are described in detail in this paper. General considerations and design specifications are introduced. The basic principles of preliminary design are discussed. Loading conditions, methods used, and results obtained in the final analysis are presented. Some important aspects in the design of details are also discussed.

INTRODUCTION

The Mackinac Straits Bridge, with its 5 miles long crossing and 100 million dollars total cost, ranks among the major bridge projects in recent times. Its main feature is a suspension bridge with 3800 ft. main span and 1800 ft. side spans (see Figure 1.). The suspension bridge with its total length of 8614 ft. from anchorage to anchorage is now the longest in the world while its main span of 3800 ft. is now the second largest. (Outranked only by the 4200 ft. Golden Bridge which may in turn soon be outranked by the Narrows Bridge.) The Mackinac Bridge has been under study by engineers for half a century. The most recent one (in 1950) was made by the Board of Engineers consisting of Messrs. O. H. Ammann, D. B. Steinman and G. B. Woodruff. Based on their report, Dr. D. B. Steinman was retained by the Mackinac Straits Bridge Authority in January 1953 to design and supervise construction of the bridge and Mr. G. B. Woodruff was engaged by Dr. Steinman as consultant.

The various features of the bridge such as history, geology, general design and layout, aerodynamic considerations, financing, construction etc., have

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been published in various articles by Dr. Steinman, Messrs. Gronquist and Boynton.[1-5] This paper is to describe the various special design considerations made in connection with the main towers of the Mackinac Bridge.

Before going any further, it might be worthwhile to point out that there are several unique features of this bridge. (i) The length of its side spans exceeds those of any existing suspension bridge. (ii) It is designed for 50 psf wind pressure on twice the exposed effective area of one cable and 1-1/2 times the area as seen in elevation on superstructure. (The Golden Gate Bridge was designed for 30 psf wind on suspended structure plus 50 psf on towers. Other existing suspension bridges such as the George Washington, Walt Whitman etc., were all designed for 30 psf.) (iii) The comparatively narrow (4 lanes) and light roadway (major portion consisting of gratings instead of solid concrete slab) for such a span length is not usual. (As it is well known to suspension bridge designers, it is easier to design for stability for a heavier bridge than for a lighter one.) How the above factors are going to affect the design of the towers will be discussed later.

It is common knowledge that a suspension bridge consists of 4 main units, namely the cables, the stiffening system, the main towers and the anchorages. It was the writer's good fortune to be responsible for the design of the main towers from the very beginning. The writer was working under Mr. R. M. Boynton, Dr. Steinman's Associate Engineer in charge of substructures and towers and his assistant, Mr. Adam Werth. With the bridge scheduled to be completed and opened to traffic in November 1957 and to be formally dedicated in June 1958, it is perhaps not untimely to present this paper now to the members of the Society for their judgment and appraisal.

Some good references can be found in the analysis and design of the suspension bridge towers. For stress in the longitudinal direction (longitudinal with respect to the bridge) Dr. Steinman's book and Mr. Birdsall's paper[6-7] give an adequate background. For stress in the transverse direction, there are papers written by Mr. Ellis and by Mr. Paine[8-10] in connection with the design of the Golden Gate Bridge. (The make up of the towers of the Mackinac Bridge is somewhat similar to that of the Golden Gate Bridge.) Although the basic considerations are essentially the same and the methods used in the final analysis are essentially classical, these methods are not the same as found in the above references.

It may be criticized that a few of the assumptions made in the final analysis seem to be too approximate. However, they may be excused by either of the following considerations: (i) that there are no better assumptions known or (ii) that the final design had to be completed in a period of approximately 6 months. It has been mentioned that Dr. Steinman was engaged by the Mackinac Straits Bridge Authority in January 1953 to design the bridge. Unfortunately the bonds were not sold until February 17, 1954, and the final design was not started until after bids for them had been received on December 17, 1953. To meet the scheduled completion of the bridge in 1957, the drawings of the towers, which had to be erected first, were to be completed by the end of June 1954, thus the total time available for making the final design computations and preparing drawings of the towers was only 6 months.

General Design Considerations and Specifications

Each tower (for general arrangement see Figure 2) consists mainly of two

steel shafts, 523.5 ft. high, resting on two concrete cylindrical shafts of 38 ft. diameter and 38 ft. high, which in turn rest on a cylindrical cassion, 116 ft. in diameter extended about 200 ft. to the bedrock. (Pier 19, sunk through 140 ft. of water and 66 ft. of overburden, a total of 206 ft.; Pier 20, sunk through 96 ft. of water and 105 ft. of overburden, a total 201 ft.) The idea of using two separate shafts 38 ft. diameter, 38 ft. high (from El. - 8 to El. + 25) was to minimize the effect of ice pressure on the 116 ft. diameter piers.[4] The two main shafts of the towers are braced at various intervals to form a rigid frame for resisting transverse wind forces.

The height of the towers were dictated by the general layout taking into consideration (i) the necessary cable sag (350 ft.) (ii) the depth of the stiffening trusses (38 ft. c.c. chords) (iii) the required channel clearance (135 ft. above mean water level for a width of a 3000 ft. channel) (iv) the width of the roadway (54 ft. clear between tower legs) and (v) the center to center distance of the cables (68 ft.). The total height of the tower shaft including the 38 ft. concrete portion is 561.5 ft. which is about the same as that of the George Washington Bridge towers (565 ft.).

The tower shafts are cruciform in shape of cellular cross section (see Figure 3) which is commonly used form for suspension bridge towers. (Notable exceptions are the George Washington Bridge which has trussed towers and the Walt Whitman Bridge which has tower legs of rectangular sections.) The main advantages of the cruciform are: (i) the maximum stresses in the longitudinal and the transverse direction do not add to each other as when stress in one direction is maximum (at extreme fiber) the stress in the other direction is low (near the neutral axis). (ii) In revising the cross section, moment of inertia of one direction can be changed without substantial effect on the moment of inertia of the other direction, as the material contributing most to the moment of inertia of one direction is close to the neutral axis with respect to the other direction. (iii) Although the rectangular cross section has larger moment of inertia in both directions than the cruciform, the large moment of inertia in the longitudinal direction, when exceeding certain limits, may not be advantageous (see discussion in Preliminary design).

The cells were made of plates and angles as shown in cross sections (see Figure 3). The dimensions of the cells were determined by the necessary working space inside. The basic cells were 3' - 8" x 3' - 8" (Golden Gate 3' - 6" x 3' - 6"). Due to the taper of the shaft, the minimum size of closed cells was made 2' - 0" x 3' - 8". The minimum thickness of the plates was specified as $\frac{1}{40}$ of the width tip to tip of connection angles. Thus 3/4" plates were used for most of the basic cells with 8" connection angles. While this is somewhat more liberal than the AASHO Specification ($\frac{1}{40}$ of the width between the nearest gage lines of angles) it is considered safe, as the tower is mainly subjected to bending stress due to wind rather than direct compression.

The splices were located by the following considerations: (i) the available size of the plates and facilities for shop assembly and (ii) the limit weight of lift during erection. The bottom three sections had to be erected with floating derrick with a limiting weight of 48 tons and the upper stories were erected by creeper traveler on the side of the tower with limiting weight of 60 tons. The distance between splices varied from 16 ft. to 42 ft., mostly around 36 to 40 ft.

One leg of each tower is provided with a service elevator and each compartment (or cell) of each leg is provided with a ladder (see Figure 3). At frequent levels man holes provide access to all compartments to facilitate access for inspection and painting. Ladders and man holes are also provided in various compartments of the struts.

The material of the towers is of carbon steel except that of the anchor bolts which is of high strength Silicon Steel. The following allowable unit stresses were specified for carbon steel:

Stress from Dead Load, Live Load and Temperature

Direct Stress only . . . . . 14 ksi
Combined direct stress and bending . . . 18 ksi

Stress from above sources combined with wind, 30% increase was allowed i.e. a maximum of 23.4 ksi.

The above allowable stresses are comparable to those specified for the Golden Gate Bridge. It should be noted that there is no ℓ/r (length to radius of gyration ratio) reduction for the allowable stress. This was because of the following reasons: (i) The axial compressive stress was low (8-12 ksi). (ii) In the direction transverse to the roadway, the ℓ/r of tower leg between struts was very low. (<25) (iii) In the longitudinal direction, the factor of safety against buckling was ample. (See later discussions.)

The towers were of riveted construction rather than welded construction. This was due to the following considerations: (i) There would be no saving in material by using welded construction, because the tower is mainly subjected to compression. (Thus only gross section needs to be considered rather than net section.) (ii) For welding construction critical joints have to be field welded and it is rather difficult to perform and inspect the field welds under precarious conditions (say 500 ft. above deep water). Primarily for the second reason, the whole bridge is a riveted structure.

The towers were designed by the well established elastic theory rather than the recently developed plastic theory (limit design), for the following reasons: (i) Since the moment capacity of a section is controlled by various riveted details, the full plastic moment of the section cannot be developed. No theoretical or experimental basis for evaluating the plastic moment capacity of such details has been established. (It seems to the writer that up to now there is no acceptable method of evaluating the plastic moment capacity of a particular type of joint except by actual testing. When various types of joints appear in a major structure, this becomes almost impossible.) (ii) The plastic method of design may not be much simpler than the elastic method due to the fact that the "Method of Superposition" can not be applied to the plastic design. (iii) Disregarding the advantage of utilizing the plastic moment capacity of the sections, so far as design for transverse wind force is concerned, there will be practically no difference between elastic design and plastic design. For example, take a simple rigid frame as shown in Figure 4. If columns AB and CD are of the same prismatic section, and suppose that the strut AC will not fail, then according to the plastic theory, the lateral load P_h at failure will be determined by a "panel mechanism" and $M_a = M_b$. This is equivalent to the assumptions that the points of inflection are located at mid-points of the columns. Of course, one would say this is not the same as the elastic theory, according to which $M_a < M_b$ unless the strut AC is infinitely rigid. However, this is only a simple rigid frame. Therefore, one

would use prismatic columns. In a major structure, it is seldom done this way. Instead one would taper the columns to make the resisting moment at A equal to M_a and that at B equal to M_b. In so doing, the plastic design and the elastic design virtually become the same. (The same situation occurs with haunched sections. See discussion in Ref. 11.) The tower shafts are tapered shafts which coincide surprisingly well with the required moment condition (see stress sheet Figure 6 and discussions later). Needless to say, it would look ridiculous to use prismatic shafts for such high towers and it is practically impossible to change sections too frequently such as to reduce sections at inflection points.

Preliminary Design

That a good preliminary design is of utmost importance is of common knowledge to every designer of statically indeterminate structures. Analysis cannot begin unless dimensions and cross sections of the structure have been predetermined. Also it is impractical to carry out lengthy and tedious analysis too often, for there may be many revisions. The best preliminary design, of course, is that in which the first assumptions become also the final design which has been accomplished on rare occasions. It may not be too much of a problem for an experienced designer to give a sufficiently good outline of a structure before starting to analyze it and to make a few revisions to reach a final design. In the present case, the first preliminary design was made by the Board of Engineers. The final design was a revised version of this first design. However, it may be helpful to the inexperienced to outline certain basic principles in obtaining the general dimensions and cross sections of a suspensions bridge tower.

First, consider the dimensions of the tower shaft in the longitudinal direction with respect to the length of the bridge. The width at top is determined by the necessary length of the tower saddle. The following basic considerations may be helpful in determining the width of the other sections.

The tower in the longitudinal direction acts as a cantilever beam subjected to axial compression and with a fixed deflection at top. (See Figure 5.) The fixed deflection is determined by motion of the cable due to live load occupying either side of the tower and due to lengthening or shortening of the cable with the rise or fall of temperature. In making the computation of the stresses in cables and stiffening trusses, assumption is made that the tower is hinged at base. In other words, the rigidity of the tower is insufficient to restrain the motion of the cable. That this assumption is substantially correct can be seen from the following data.

For the tower as designed, with fixity at base, to deflect the top of each tower leg, a horizontal force (F) applied at the top is required. The value of F for 1 ft. deflection is only 66 kips (F_1). To deflect the same point, if the tower is hinged at base, it requires an increase in the horizontal component (H) of the cable stress. The increment of H for 1 ft. deflection is approximately 1250 kips (H_1). (Proportioned from the condition of live load in main and far side spans only with no temperature change.) Assuming that the tower is hinged at base, the maximum deflection of 3.08 ft. toward the main span was obtained. With the tower fixed at base, to obtain the same deflection, it requires that the cable H in the side span be smaller than the H in the main span by an amount equal to F ($\delta H = -F$) of approximately 29 kips ($0.44\ F_1 = 0.44 \times$

66), a negligible amount with respect to total H. It means that the modification of the deflection due to tower rigidity amounts to only

$$\frac{\delta H}{H_1} = \frac{29}{1250} = 0.023 \text{ ft.}$$

(i.e. corrected deflection = 3.08 - 0.02 = 3.06 ft.).

Since the tower acts as a cantilever beam with a fixed deflection (Δ) at top, the deflection (Δ) can be obtained by taking moment of the $\frac{M}{EI}$ diagram about tower top. (Where M is the bending moment and E is the modulus of elasticity and I is the moment of Inertia at the section considered.) Since $f_b = \frac{Mc}{I}$ where f_b is the fiber stress due to bending and c is the distance from neutral axis to the extreme fiber), one may also find Δ by taking moment of the $\frac{f_b}{Ec}$ diagram. Consider simply the case of a cantilever beam of length L and of uniform section (c = constant c_o) and make f_b = constant f_{bo} (which should give the most effective design). One finds that:

$$\Delta = \frac{f_{bo}}{E c_o} \times L \times \frac{L}{2} \tag{1a}$$

therefore
$$f_{bo} = \frac{2 E c_o \Delta}{L^2} \tag{1b}$$

and
$$c_o = \frac{f_{bo} L^2}{2 E \Delta} \tag{1c}$$

Assume that a portion of the allowable stress (f) be assigned to bending stress f_b. (This value, of course, should be larger for large value of Δ and vice versa. The writer would arbitrarily take $f_b \leqslant \frac{f}{2} = 9$ ksi for economical design.) Taking f_b = 8 ksi, L = 520 ft., E = 29 x 10^3 ksi and Δ = 3.08 ft. one finds that the necessary width of the cantilever of uniform cross section $d_o = 2c_o$ = 24 ft. This at least gives the order of magnitude of the longitudinal width of the tower leg. (The actual width is as follows: 15 ft. at top tapered for 350 ft. to 23 ft. then with 146 ft. parabolic flare out to 30.5 ft. at base. The actual average width is about 21 ft.) With the width at top known and the required average width found, one may proceed to draw the outline of the tower in longitudinal direction with due regard to aesthetic appearance.

From formula (1b.), one can see that f_{bo} is actually proportioned to c_o, and one may then wonder whether the tower width should be further reduced for reducing the stress f_b. Three points must be considered in this connection, the first two having already been mentioned, namely (i) the necessary width of the top is fixed by the required length of the saddle (ii) the necessary width from the aesthetic point of view and (iii) the necessary rigidity of the tower shaft. Point (iii) merits exploiting further. The question is: What is the necessary rigidity of a suspension bridge tower shaft?

In the writer's opinion, the minimum rigidity should be such that in addition to the deflection due to vertical cable reaction and tower weight, a positive force F (see Figure 5) is required to bring the tower top to its maximum deflected condition under dead load, live load and temperature (D + L + T). It

is not uncommon that suspension bridge towers have been designed with (-F). Such a tower is unstable as a free standing cantilever column subjected to axial compression imposed, and requires the cables to serve as guys to hold it back to prevent further deflection. That this condition is not in itself unsafe can be seen from the fact that rocker towers which are hinged at base have been in use.(6) The rocker tower shaft, being unstable under all conditions as a free standing column, acts as a column with simple supported ends whose required rigidity, in case of a column of uniform cross section, can be easily provided by limiting the ℓ/r ratio. The rocker towers are not used for major suspension bridges because of difficulty in erection. The stiffness of a tower shaft fixed at base, however, can be best understood by comparing it with that of the rocker tower. Consider a column of length (ℓ) with uniform cross section fixed at base and simply supported at top. Let the elastic buckling load of such a column be Pue, that of a rocker tower of the same length and cross section be Pre and that of a free standing cantilever of the same length and cross section be Pce. Since the reduced length is 0.7ℓ for Pue, ℓ for Pre and 2ℓ for Pce,(12) we have

$$P_{ue} = \frac{P_{re}}{(0.7)^2} = 2P_{re} \quad \text{and} \quad P_{ce} = \frac{P_{re}}{(2)^2} = \frac{P_{re}}{4}$$

Therefore Pue = 2 pre = 8 Pce. However, the actual factor of safety is not that big when the value of ℓ/r is moderate. At $\ell/r = 50$, for example, the actual plastic buckling load of a simply supported column of carbon steel is only about 1/4 of the elastic buckling load. (The plastic buckling load is approximately equal to the elastic buckling load at $\ell/r = 100$). Therefore, the plastic buckling load for a uniform column of carbon steel fixed at base and hinged at top is approximately $P_u = 2 \times \frac{Pre}{4} = 2$ Pce. When F = 0, the axial load is equal to Pce, the elastic buckling load of a free standing cantilever. When F is positive and $\ell/r \geqslant 50$, the actual factor of safety against buckling is better than 2. The actual suspension bridge tower shaft is not of uniform cross section. The actual plastic buckling load may theoretically be figured out according to the tangent modulus theory. However, this will be a lengthy tedious process. The above discussion serves to illustrate that if F is positive under the condition D + L + T, ample factor of safety against buckling of the tower shaft as a whole will be assured. (Note: F may be negative under wind. However, this is due rather to lateral bending than to buckling.)

The computation of whether +F or -F is required of course, cannot be carried out while barely knowing the outline in the longitudinal direction. The area and moment of inertia of the sections have to be known first and they cannot be fixed without considering the requirement in the transverse direction.

Given the center to center distance of the cable, the necessary clear width between tower legs, the height of the tower and the roadway level, the struts necessary for providing rigid frame action are then arranged. The wind pressure acting on the bridge is carried in part by the cables and in part by the lateral system of the stiffening trusses.(13) The wind force acting at top of the tower comes from the reaction of the cable and that acting near the roadway level comes from the reaction of the lateral system. The wind pressure acting on the tower in the transverse direction also is known, as the

width of tower in longitudinal direction has been tentatively determined.

Points of inflection can then be assumed in the tower legs between struts. They are assumed somewhat above the midpoint between struts as the stiffness of the struts and the moment of inertia of the legs are inevitably smaller at top than at bottom. The points of inflection of the struts are always at their midpoint for a symmetrical frame subjected to lateral load. Moments, shears and axial forces in the tower legs and the struts can then be computed.

Assuming a certain portion of the allowable stress as the axial stress, the required sectional area at several critical levels can be determined. Meanwhile the required sectional area of the tower top is determined by the maximum vertical reaction at top divided by the maximum allowable axial stress (14 ksi). In the case of the tower designed, the portion of the allowable stress taken up by the axial stress is about 12 ksi. That this is higher than the 10 ksi axial stress for the longitudinal direction is due to the additional axial compression caused by wind in the leeward leg. The remaining portion of the allowable stress (1.3 x 18 - 12 = 23.4 - 12 = 11.4 ksi) is due to the bending moment caused by wind and temperature with the wind taking up about 80% of the stress. Thus with the bending moments and the allowable stress known the required section modulus at critical levels can be determined. [The proportion between direct stress and bending stress, of course, should depend on the following: (i) the type of the tower construction (for example, bending stress is rather small for the trussed type) and (ii) the magnitude of the lateral force (for example, direct stress should be higher and bending stress lower if 30 psf wind is specified instead of 50 psf.)]

Knowing the required area and section modulus, the general shape of the cross section and the width in the longitudinal direction, a tentative area of a basic cell can be assumed and tentative arrangement of the cells can be made to suit the above conditions. This also determines the required width in the transverse direction.

With the shear in the struts known, the required area of the chords and diagonals of the struts can be determined approximately. However, it should be noted that stresses in these members should be deliberately kept low. This is because (i) struts should be designed for stiffness(8) as well as strength and (ii) considerable secondary bending stresses are expected to develop in the short stocky strut members. In connection with stiffness (or flexibility) of the struts, it may be of interest to point out that, for the tower designed, the stiffness of a strut due to deformation of the diagonals (shear deformation) is only about 1/2 as much as that determined by considering the moment of inertia of the strut alone.

Knowing the cross section widths in longitudinal and transverse directions for critical points, the outline of the tower can be drawn and its appearance should be examined very carefully from the aesthetic point of view.

Studies should then be made of the possible arrangements of details such as joints, splices, diaphragms, etc. The base dimensions of the tower should also be examined carefully in case there is any difficulty in possible arrangements of anchor bolts.

With the first preliminary design thus made, quick but more accurate and detailed analyses should be made for the stresses in both the longitudinal and the transverse directions to see if there are any necessary major revisions of sections and dimensions. Several revisions may be necessary before a final design is reached.

Final Analyses and Resulting Stresses

After the dimensions and sections had been decided as final, careful analyses were carried out to review the design. The following presentation includes: loading conditions considered, methods used, and results obtained.

I. Stresses in the Longitudinal Direction

A. Loading Conditions

1. Dead Load plus Live Load and Temperature (D + L + T)
   L.L. 1500 lb. per Lin. ft. of bridge (see later discussion)
   Temperature $\pm$ 60°F
   Deflection at top:

Normal position of tower	- 0.25'		- 0.25'
Deflection of tower	+ 3.12'		- 2.66'
Allowance for error in cable stringing	+ 0.21'		- 0.21'
total	+ 3.08'	(governs)	- 3.12'
Cable reaction per leg	20,850k		19,390k

2. Dead Load plus 50 psf Wind (D + W_{50})
   Tower free standing - erection condition
   The magnitude of the force due to wind pressure on tower in the longitudinal direction can be seen from the fact that it causes a shear of 644k and a moment of 160,000 ft. k. at base of each tower leg.

3. D + W_{50} - T
   Deflection at top 1.63 ft.

4. D + L + T + W_{25}
   Deflection at top 3.37 ft.

It should be noted that, as it can be seen from the stress sheet (Figure 6 b.) that loading condition (1) is the most critical one. The tower under free standing condition, loading condition (2), is critical for designing the anchor bolts at base.

Several points should be noted with regard to the loading condition (1). (i) Under full dead load and normal temperature the tower is deflected 3 inches shoreward, for the purpose of balancing the riverward and shoreward deflections under live load and temperature. (ii) The bridge was designed for a live load of 2000 lb. per linear ft. of bridge. However, for tower deflection, 75% of that amount (1500 lb. per ft.) was used. This is because of the fact that unbalanced loading—live load on one side of the tower and none on the other side—of the full designed amount (2000 lb. per ft.) is extremely improbable. (iii) Even with 75% of the design load, the theoretical deflection of tower top of 3.08 ft. exceeds that of any of the existing suspension bridges (for example, 22 inches for the Golden Gate Bridge.) This is because the Mackinac Bridge has a ratio of main span to side span of approximately 2 : 1 and has established a new record of side span length (1800 ft.).

B. Method of Computation and Resulting Stresses

The method used in computing the longitudinal stress is not new in principle. The method consists essentially of the following steps: (i) Compute the deflection curve due to a unit horizontal force applied at top. (ii) Assume a deflection curve which gives the required deflection (Δ) at top.

Compute the bending moment induced by this deflection curve due to cable reaction at top and weight of tower at various levels. Compute the resulting deflection at top (Δp). (iii) Compute the value of $\Delta - \Delta_p = \Delta_F$. Find the corresponding horizontal force F to produce the Δ_F from the results of step (i). Superimpose the deflection due to vertical forces and the deflection due to F to get the resulting deflection curve. (iv) Check this deflection curve with the assumed one. If they agree, compute the resulting direct and bending stresses. If they do not agree, revise the assumption and repeat the procedure.

Since the tower section is not uniform throughout, numerical integration method (or commonly known as area moment method) has to be resorted to. The integration procedure adopted was the one developed by Professor Newmark.(14) It is probably the first time that this procedure has been applied to the design of major suspension bridge towers. In finding the deflection curve, the metallic tower above the concrete base was divided into 20 equal divisions. The 33 ft. long 38 ft. diameter concrete shaft from the tower base down to the 116 ft. diameter caisson was considered as one division with equivalent transformed area of steel. The tower was considered as fixed at the base of the concrete shaft.

The resulting moments and stresses are as shown in the stress sheet (Figure 6a and b). The direct stress is indicated as P/A and the bending stress as Mc/I in the diagram. The efficiency of the design can be seen from the closeness of the stress curve of the loading condition (1) with the limiting allowable stress of 18 ksi. The stresses in combination with wind do not govern because of the higher allowable value.

II. Stresses in the Transverse Direction

A. Loading Condition

The stresses in the transverse direction were principally designed for the loading condition of dead load plus wind load (50 psf) and temperature ($D + W_{50} + T$) with the following data.

Vertical reaction on tower by cable	18,800^k/leg
Horizontal reaction on tower by cable	858^k/leg
Horizontal reaction on tower by lateral trusses	731^k/leg

Temperature difference between tower and base at Elev. -8 $\pm 35^oF$
Temperature difference between struts $\pm 10^oF$

The loading due to wind is shown in Figure 7 (a). The wind pressure on the tower is considered as distributed load in the analysis. The magnitudes of the resultant forces at various levels can be seen from the shears due to external loads as shown in Figure 7 (d). The resultant wind pressure on each tower leg (not including the concrete shaft) is 542 kips applied at a distance of 223 ft. above the base of the steel shaft.

B. "Primary Stresses" due to Dead Load and Wind

It should be made clear in the beginning that the definition of the "Primary Stress" (with quotation marks) as used here is somewhat different from that which is commonly understood in structural engineering due to the failure of finding more proper wording. "Primary Stress" simply means that it contributes to a major portion of the resulting combined stress. "Secondary Stress" (with quotation marks) simply means that the stress is of comparatively minor importance. Secondary stress (without

quotation marks) has the usual meaning that the stress contributes nothing to the load carrying capacity of a structure such as the bending stress in members of a simple statically determinate truss caused by joint rigidity. The following stresses are considered as "secondary" in the subsequent discussion: (i) the addition bending stress due to eccentricity of the dead load caused by the deflection of the tower leg under the action of lateral wind, (ii) the additional bending stress due to the eccentricity of axial forces (dead load and wind) resulting from the fact that the assumed axis of the leg does not pass through the centroids of sections at all levels, (iii) the additional bending stress due to the weight of struts, and (iv) bending stresses due to the inclination of the column axis under dead load.

For most sections, the combination of the "primary" wind stress plus "primary" dead load stress constitutes about 90% of the total stress. The "primary" dead load stress can be computed simply and directly. Therefore, only the method of analysis of wind stresses needs to be discussed.

An inspection of the framing reveals that the method of moment distribution cannot be adopted for accurate analysis. The reason is that to use this method would imply the neglect of the following effects: (i) axial deformation in the members of the K truss, (ii) axial deformation in the legs, (iii) axial deformation in the struts, (iv) shear deformation in certain parts of the legs (mostly at the struts), and (v) shear deformation of the struts, although this last effect may be taken care of by finding the equivalent stiffness of the strut by including the shear deformation. However, in obtaining certain minor stresses such as those due to axial eccentricity of dead load, the moment distribution method was used.

One may use the modified slope deflection method as developed by Professor Ellis.[8] However, it is the writer's opinion that it would make the problem unnecessarily complex to include many unknowns in order to solve the primary and secondary stresses all at once as it was done by Professor Ellis.

In order to solve the problem as simply and directly as possible and to include all effects of axial and shear deformations, the classical work method (it has several other names such as strain energy method etc., but will be referred to hereafter as the work method) was used. (There are many references on this method, to cite just one, Ref. 15.)

The tower frame was broken up into 5 frames and 5 unknown moments ($X_1 \ldots X_5$) were introduced as redundants as shown in Figure 7 (b). It should be noted that under lateral wind load, the resulting deflection of the frame will be antisymmetrical. Thus the points of inflection will be at the mid-points of all struts. Without redundants, or if the redundants are known, the structure becomes statically determinate. A set of 5 equations can be written with the following general form.

$$\sum_j X_j \delta_{ij} + \delta_{io} = 0 \qquad (2)$$

where δ_{io} is the deformation at point i (point 1, 2 . . or 5) due to the external load (with the modulus of elasticity E factored out), δ_{ij} is the deformation at point j (point 1, 2 . . or 5) due to the unit moment applied at point i (point 1, 2 . . or 5),

$$\delta_{io} = \int \left[\frac{M_o m_i}{I} + \frac{E}{G} \frac{V_o v_i}{A'_w} q + \frac{U_o u_i}{A} \right] ds \quad (3a)$$

$$\delta_{ij} = \int \left[\frac{m_i m_j}{I} + \frac{E}{G} \frac{v_i v_j}{A'_w} q + \frac{u_i u_j}{A} \right] ds \quad (3b)$$

where the integral sign implies numerical integrations

ds is the increment of the length of a member

m_i, v_i, and u_i are respectively moment, shear, and axial force due to the unit moment applied at point i. m_j, v_j, and u_j, those due to the unit moment applied at point j. (See schematic diagram shown in Figure 7c.)

M_o V_o and U_o are respectively moment, shear and axial force due to external load acting on the five frames (see diagram in Figure 7 d.)

A, is the cross sectional area; A'_w is the area of the 3 web plates of the strut chords, or the area of the 3 main web plates in the transverse direction of the tower leg; I is the moment of inertia in the direction of bending. (All quantities are those at the same point where moment, shear and axial stress are computed.)

E/G is the ratio of the modulus of elasticity to the shear modulus.

q is a shape factor applied for correcting the shearing stress computed from the area of main webs only. (See above for definition of main webs.)

The final moment, shear and axial force in the members are given respectively by the following equations:

$$\begin{aligned} M &= M_o + \Sigma X_i m_i \\ V &= V_o + \Sigma X_i v_i \\ U &= U_o + \Sigma X_i u_i \end{aligned} \quad (4)$$

Due to the change of cross section, the center of gravity axis of the section becomes zigzag. The distance between axes of the tower legs was taken as 68.6 ft. at top, widened to 69.2 ft. at point 1 in a distance of 110 ft., to 71.0 at point 2 in a distance of 124 ft., to 71.8 ft. at point 4 in a distance of 180 ft., to 72.8 ft. at the base of the steel section in a distance of 103 ft., then changed to 70 ft. to agree with the center to center distance of the 33 ft. long concrete shafts. The axes were taken as straight lines between those points (see Figure 7 b).

Assumption was made that the members of the strut K were hinged at their ends. Although this is not true, the ratio of the stiffness of these members to the stiffness of the tower leg is so small that the secondary moments in these members would hardly have any effect in the stress in the tower leg. The eccentricity of the connection of the diagonal of K at point 4 was taken into consideration in the analysis.

Each of the 4 struts, exclusive of the K strut, is a statically indeterminate structure in itself. However, in analyzing the stresses of the tower as a whole, simplifying assumptions seem to be justified in figuring the deformation of these struts. The chords of the struts, their rigidity being far greater than the rigidity of the web members, were considered as beams capable of resisting shear and bending. The diagonals were assumed to be hinged to the chords and to carry equal amounts of shear. The verticals were assumed as idle. Under the action of either the external

wind force or the internal redundent moment, the force resisted by each of the struts is simply a vertical shear applied at the inflection point at the mid-span (see Figure 7 c). Based on the aforementioned assumptions it was found that the diagonals carried about 66% of the shear, while the upper chord carried 12% and the lower chord 22%. With the shear distribution known, the contribution of the chords and diagonals to the strut deformation can easily be computed.

Another problem in connection with the deformation of the struts is whether one should assume an extension of strut to the axis of the tower leg. If such an assumption is to be made, what should be the section of the strut for the portion extended inside the column. While it makes practically no difference whether such extension should be assumed inside the column when the width of the column is small, it makes some difference when the distances from the face to the axis of the column vary from 7.33 ft. to 9.00 ft. The argument for assuming the extension runs as follows: The integral $\int \frac{Mm}{I} ds$, for example, implies that ds should be taken along the axis. The assumption of no extension implies that the I of the strut jumps to infinity immediately inside the face of the column. That means the stress is being reduced to zero immediately, can this be true? It was finally decided that it would be conservative to consider an extension of strut inside the tower leg to the axis of the leg. The extension was considered as consisting of 3 column web plates about 36 ft. deep vertically. These plates are continuations of the strut web plates and have the same depth as the extension of the strut to column axis. The maximum difference between the above assumption and the assumption of rigid joint (no deformation at the joint of column and strut) was found in the moment of X_3. This moment computed with the above assumption was 13% higher than the same computed with the assumption of rigid joint, corresponding to a difference in stress of about 1 ksi.

The resulting moments and stresses are as shown in Figure 6 (c) and (d) in dotted line. The fact that the point of inflection of the lowest frame is near the bottom chord of strut A (see Figure 6c) may seem to be strange. However, it has been found that this is primarily due to three effects: (i) the effect of axial and shear deformation in the tower legs which is usually neglected in the ordinary method of analysis, (ii) the 1.4 ft. offset at the tower base between the axis of the tower leg and the axis of the concrete shaft, and (iii) the moment of X_4.

C. Temperature stress and "Secondary Stresses"

1. Temperature stresses were analyzed by both the work method and the moment distribution method. In using the work method for the whole structure, however, 10 unknowns need to be solved. This seems to be unjustified for the magnitude of the stress involved. Sufficiently good approximation can be made by neglecting the effect of the frame further from the strut considered in expansion or contraction. For example in considering the expansion of strut A, the portion of the frame above K may be entirely neglected.

2. The additional bending stresses due to eccentricity of the dead load caused by the deflection of the tower leg under the action of lateral wind were computed. The axial force of dead load includes the dead load reaction of the cable and tower weight at various levels. This bending

effect was considered in analyzing the stresses in the tower of the Golden Gate Bridge by slope deflection method.[8,9] For the present case, the following method was used: (i) From the "primary" bending moment already found, compute the deflection curve of the tower leg by means of moment area method. The deflection amounts to 1.35 ft. at top which is not a negligible amount. (ii) From the deflection curve, find the moment acting on each of the 5 frames under the action of dead load. (Note that the axial forces in the tower legs caused by wind, being compressive in one side and tensile in the other side, produce no net moment on the frames.) (iii) From these moments, which are rather similar to the wind moment, compute the δoi, solve for $X_1 \ldots X_5$ and find the resulting moments (M'). (iv) Since these moments would cause additional deflection and therefore additional moments, the final moment (M") is given by the following formula:

$$M'' = \frac{M'}{1 - \frac{M'}{M}} \qquad (5)$$

Where M is the "primary" wind moment found previously. The final top deflection was estimated to be 1.44 ft.

3. The additional bending moments due to the eccentricity of axial forces (dead load and wind) resulting from the fact that the assumed axis of the leg does not pass through the centers of gravity of sections at all levels were computed by the moment distribution method. The additional bending moments due to the weight of the struts were computed by the same method.

4. Due to the small inclination of the column axis, under the action of dead load, there will be compression in the struts. The shortening of the struts, like the effect of temperature, will induce bending moments in the column legs. However, it was found that this effect is extremely small.

D. The Resulting Moments and Stresses

The resulting moments and stresses are as shown in the diagrams of Figure 6 (c) and (d) in solid lines. It can be seen that the combined effect of the temperature and "secondary" stresses are relatively small in comparison with the combination of "primary" stresses due to dead load and wind. The proximity of the peak stresses to the maximum allowable stress of 23.4 ksi proves that the sections are well chosen.

The Design of Details

I. The Base Plate and the Anchorage

The base plate is 5 inches thick with dimensions as shown in Figure 8. In designing, the plate was considered as fixed at the edges which were taken as center lines of the plates of each cell. (See for example, shaded portion in plan view of Figure 8.) Based on reference (16) the design moment was taken as the average of the maximum moment at edge and the maximum moment at center (such as the average of the moments at points a and b in the aforementioned shaded portion).

The two most critical loading conditions for the anchorage design are: (i) Tower free standing subjected to 50 psf wind in the longitudinal direction during erection and (ii) dead load plus 50 psf wind plus temperature effect in the transverse direction under service condition.

One usual method of design of the anchor bolts is to analyze the section below base plate as if it were a reinforced concrete section with the anchor bolts acting as tension steel. However, such assumption is open to question since the anchor bolts do not transmit stress immediately into the tower leg section. The bolts were projected 4'-10" above the base before transmitting their stresses through bearing plates on angles connected with tower section (see Figure 8). Thus in order to stress the anchor bolts to their full allowable values, there must be an opening in the tension side equal to the stretch in the distance of 4'-10". Although the opening is small (about $\frac{1}{16}$"), it is not infinitesmal. Of course, this does not endanger the safety of the tower. What it means is (i) that the assumption of continuity of the steel section with the concrete shaft becomes invalid and (ii) that there will be some yielding of steel and high bearing stress in concrete in the compressive side. The question is: are these desirable?

The other method of design is to prestress the bolts such that there will be absolutely no opening under any loading conditions. The final adopted design approached this objective but not quite. Under extreme loading conditions, the section between base plate and concrete shaft becomes a partially prestressed section and the maximum opening will be about 0.014".

There are 68 silicon steel anchor bolts of 4 inches diameter for each tower leg arranged in a pattern as shown in Figure 8. All bolts were placed outside the tower section to facilitate tightening and possible retightening in the future. The bolts extended 14 ft. into the concrete shaft (see Figure 8) and were prestressed to 300 kips each. To facilitate the prestressing, the bolts were set in 6" diameter pipe sleeves. To help accurate positioning of the bolts, the pipe sleeves were braced together with light shapes to make up 4 frames; (Frames I and III, one each; and 2 frames of II, Figure 8a) which in turn were braced together by another frame (Frame IV) to form a single anchor bolt frame. The spaces in between pipe sleeves and bolts were grouted after prestressing.

One might get the impression that there are too many anchor bolts for this tower as compared with other suspension bridge towers. However, the following facts should be noted: (i) That the bridge is comparatively light due to the use of grating for part of the deck, (ii) that it is comparatively narrow, and (iii) that it is designed for much higher wind load. Due to these facts, the base section of the tower may have a tension of 10.7 ksi due to bending moment under service condition which is unusual for suspension bridges.

II. The Splices for the Tower Shaft

These splices were designed according to the following specifications:

Exterior material	75% of the gross area strength
Interior material	50% of the gross area strength

That these specifications are adequate can be seen from the following facts: (i) For compression, with materials milled to bear, splice for 50% gross area strength is allowed in the A.R.E.A. Specifications. (ii) The maximum tensile

stress is found to be about 10.7 ksi under wind which is less than 50% of the maximum allowable stress of 23.4 ksi.

One inch diameter rivets were used for all splices and connections (except those in horizontal leg of diaphragm angles which were of 7/8" diameter).

III. Diaphragms and Brackets

Diaphragms spaced from 7' -1" to 11' -6" (being closer at bottom) were provided in the cells of the tower shaft. They were 1/2" plates provided with manholes and with one 4 x 3 1/2 x 1/2 connection angle on each edge. The edges were milled to keep the cells true to form.

At tower top, vertical diaphragms were provided to distribute the load from tower saddle to the tower sections.

During erection, the tower saddle had to be set back shoreward 3 ft. from its final position. Brackets were provided at tower top to support the tower saddle during erection and to jack it back to its final position. These brackets will be removed after erection.

Brackets were also provided for the hangers of the stiffening trusses to take up the reactions of the trusses.

IV. Connections between Tower Units

To facilitate the fabrication and erection, the tower shaft is divided into 4 units. Let us call them the center wing (C.W.) the transverse wing (T.W.) and the two longitudinal wings (L.W.) (see Figure 9 for division and location of each).

The connection of L.W. to C.W. was made by rivets such as "a" and "b" in the connection angles (see Figure 9). Each of these rivets resists a shear of

$$v = \frac{Q_L}{I} \frac{p}{n} V \qquad (6)$$

Where Q_L is the static moment of the longitudinal wing about neutral axis of the whole section, "I" is the moment of inertia of the whole section, "p" is the pitch in a rivet line, "n" is the number of rivet lines and "V" is the shear in the longitudinal direction.

$V = P \frac{dy}{dx}$, or $P \frac{dy}{dx}$ plus shear caused by longitudinal wind, where P is the axial load at a certain level and $\frac{dy}{dx}$ is the slope of the deflection curve at that level.

For the portion of the tower shaft between struts, the shear in rivets of the connection between T.W. and C.W. such as rivets "a," "b" and "c" can be computed by a formula similar to (6), using Q_T (subscript T stands for transverse direction) instead of Q_L and V equal to the shear caused by transverse wind.

It was found that the 4" staggered vertical pitch in general was satisfactory for both of the above design conditions.

The connection of the T.W. to C.W. for the portion of the tower shaft at struts, however, merits further study. Consider a portion of a beam as shown in Figure 10 (a) subjected to a moment M_t at top and a moment $M_b = M_t + \Delta M$ at bottom. The stress (f) distribution is given by the formula $f = \frac{Mx}{I}$

where x is the distance from the neutral axis. If a vertical section is located at a distance x_0 from the neutral axis, then from Figure 10 (a) the shear of the section is equal to $V = \int_{x_0}^{c} \frac{\Delta M x}{I} dA$ where dA is an increment of area of the horizontal section and "c" is the distance from neutral axis to extreme fiber. The moment on the vertical section at x_0 is equal to

$$M = \int_{x_0}^{c} \frac{\Delta M x (x - x_0)}{I} dA$$

Therefore
$$V = \frac{\Delta M}{I} \int_{x_0}^{c} x \, dA = \frac{\Delta M Q'}{I} \tag{7}$$

$$M = \frac{\Delta M}{I} \int_{x_0}^{c} x^2 dA - \frac{\Delta M x_0}{I} \int_{x_0}^{c} x \, dA = \frac{\Delta M}{I} (I' - x_0 Q') = \Delta M q' \tag{8}$$

Where
$$Q' = \int_{x_0}^{c} x \, dA, \quad I' = \int_{x_0}^{c} x^2 dA \quad \text{and} \quad q' = \frac{I' - x_0 Q'}{I}$$

Let the section at bottom be subject to a shearing force Vo. Then the shearing force is distributed according to $v = \frac{V_0 Q}{Ib}$ where Q is the static moment of the cross-sectional area between x = c and x = x and b is the width of the section. The total shearing force carried by the portion of the section between x = c and x = x_0 is equal to (referring to Figure 10 a)

$$V' = \int_{x_0}^{c} v \, b \, dx = \int_{x_0}^{c} \frac{V_0 Q}{Ib} b \, dx = V_0 \frac{xQ}{I} \Big]_{x_0}^{c} - V_0 \int_{x_0}^{c} \frac{x \, dQ}{I}$$
$$= -\frac{V_0 x_0 Q'}{I} - V_0 \int_{x_0}^{c} \frac{x(-x) dA}{I} = -\frac{V_0 x_0 Q'}{I} + V_0 \frac{I'}{I} = \frac{V_0}{I}(I' - x_0 Q') = V_0 q' \tag{9}$$

Take the portion of the tower leg at strut as shown in Figure 10 (b), where V_S and M_S are respectively shear and moment from the strut acting on the inner face of the tower leg; $\bar{x}$ is the distance from the center of gravity axis of the tower leg to its inner face; x_0 the distance from the same to the connection of T.W. and C.W.; $\bar{x}'$ the distance from the same to the center of gravity of the horizontal section of the T.W.; M_t, N_t and V_t are respectively moment, axial force and shear on a section at top of the strut; and M_b, N_b, V_b, those on a section at bottom of the strut; and Z_0 is the depth of the whole strut.

Assume that the wind force on the portion of tower in a distance Z_0 is negligible, then $V_b = V_t = V_0$. Assume also that the sections at top and at bottom are the same and the effect of the slope of the outer face of the tower leg in a distance Z_0 is negligible. Then the shear and moment acting on the connection between T.W. and C.W. are respectively (see Figure 10 c)

$$V = \frac{\Delta M Q'}{I} + (N_b - N_t) \frac{A'}{A} = (V_S \bar{x} + M_S - V_t Z_0) \frac{Q'}{I} - V_S \frac{A'}{A} \tag{10}$$

$$M = \frac{\Delta M}{I} q' + V_t q' Z_0 - V_S \frac{A'}{A} (\bar{x}' - x_0) = (V_S \bar{x} + M_S) \left(\frac{I' - Q' x_0}{I}\right) - V_S \frac{A'}{A} (\bar{x}' - x_0) \tag{11}$$

Where A' is the cross sectional area of the transverse wing (T.W.) and A is the area of the cross section of the tower leg as a whole.

From the above, it can be seen that, due to the existence of the moment M between T.W. and C.W., there will be tension in the connection between them. Now, there is no doubt that rivets can resist tension as well as shear. The trouble is that it will bend the connection angles and their resistance to bending is low. The two 17" x 5/8" plates as shown in Figure 9 and in Figure 3 have the function of tie plates in addition to their normal function of providing necessary area for the tower section.

V. Connection for the Struts

The struts were connected to the tower shaft normally by extending the 3 main transverse web plates of the tower leg to a distance of 1' -0 3/4" beyond the inner face of the leg and splicing the 3 strut web plates thereon by means of splice plates for the full depth of the strut (see Figure 11a). However, for struts B and C (third and second from top respectively), to relieve high stress concentration due to bending moment in the top chord, wider splice plates were used and 3 of the splice plates were extended into the tower leg for a distance almost equal to the width of the inner cell (see Figure 11 b).

Many of the joints are subjected to high bending moment, axial force and shear, such as the one mentioned above connecting the top chord of a strut to the tower leg, or the joint at the junction of the diagonals and the top chord of the K strut where bearing blocks were connected to transmit the wind reaction from lateral trusses to the tower. The details of these joints are so complicated that one can only do his best to assume the paths of stresses and the distribution of loads and moments to various parts of the connecting materials.

CONCLUDING REMARKS AND ACKNOWLEDGMENT

The weight of steel in the two main towers is approximately 13,000 tons. The total cost of the towers was approximately $7,250,000.00. The design drawings were issued in July 1954. Erection of the towers was started in July and completed in December 1955. For a picture of the completed tower, see Figure 12.

The writer is gratefully indebted to Dr. D. B. Steinman and Mr. R. M. Boynton for giving him the responsibility to work on the project and for their direction and supervision; to Mr. Adam Werth for consultation at various stages, to Messrs. H. Y. Lu, R. M. Mayrbaurl and many individuals working on the project whether their work is computation, design or drafting. May this report and the towers at the Mackinac Straits serve as testimonials of their collective effort.

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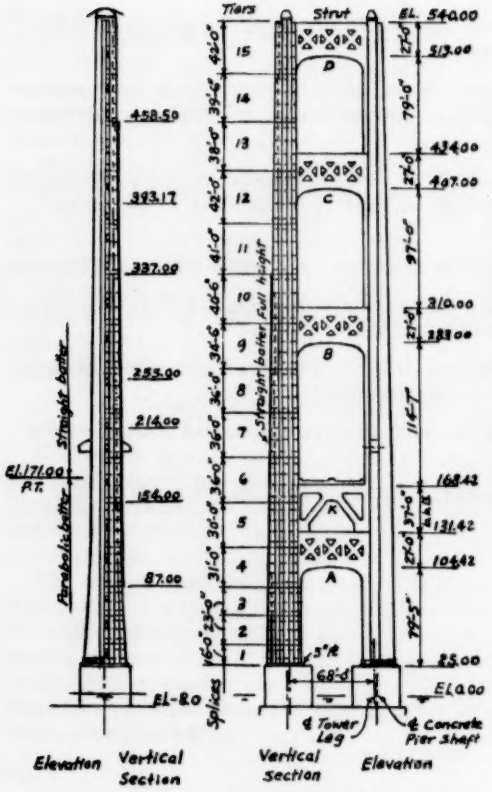


FIG 2 MAIN TOWERS
Vertical Section & Elevation

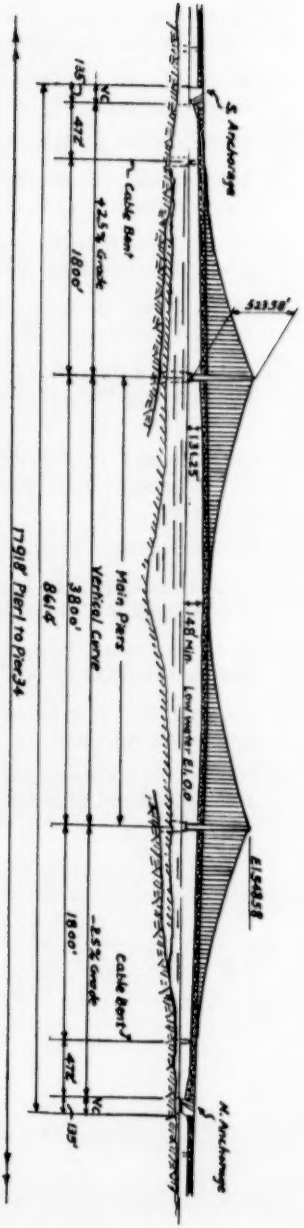
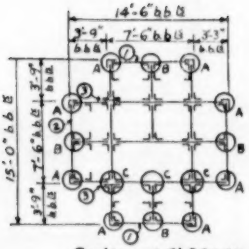
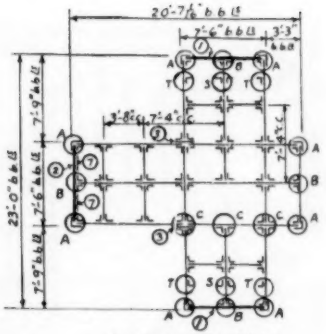


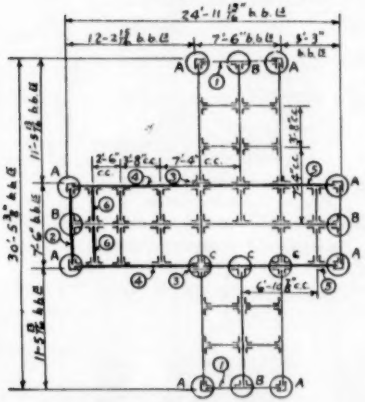
FIG. 1 MACKINAC BRIDGE -- ELEVATION



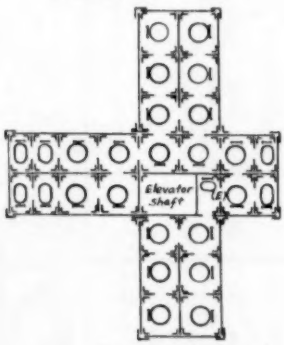
Section at EL 340.00



Section at EL. 171.00



Section at EL. 25.42



Typical Arrangement of Holes and Ladders

Shape	Location	Mark & Symbol	Notes
R 87x1⅜	Tiers 4 to 13	① Heavy Line	1. Typical Sections Shown are for main material only. Vertical diaphragms in Sections at El 340 & El 25.42 are not Shown.
R 87x⅝	Tiers 1 to 3 & 14 to 15	① Medium Line	
R 87x1⅜	Tiers 1 to 10	② Heavy Line	
R 87x⅞	Tiers 11 to 15	② Medium Line	
R 17x⅜	Full height	③ " "	
R 1⅜	Tier 1 only	④ Heavy Line	2. For Vertical Sections see FIG 2.
R 1⅛	Tier 1 only	⑤ " "	
R 42x¾	Tiers 1 & 2 only	⑥ " "	3. N.W. & S.E. Legs are as Shown with elevator. N.E. & S.W. Legs are as Shown in the upper half of Sections B. Symmetrical about ℄ (without elevator)
R 28x⅝	Tier 6 only	⑦ Light Line	
R 3⅜	All other Plates	No Mark Medium Line	
L 8x8x⅝ & L 6x6x⅝	Full height	A	
" 8x6x⅝	" "	B	
" 8x8x⅝	" "	C	4. The Symbol B in Vert. Section represents a whole set of openings to every cell of the section and O represents the emergency opening (E) in hor. Section
other L 8x6x⅝	" "	No Mark	
2 PL 8x8x⅝	As shown in Vert Sect.	····· S in horiz. Sect.	
4 PL 8x6x⅝	" " "	---- S & T in hor. Sect	
8 PL 8x6x⅝	" " "	— — —	
8 PL 8x8x⅝ & 2 Web PL ⅝	" " "	———	

FIG 3. TOWER SECTIONS

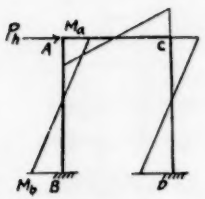


FIG. 4

FIG. 5

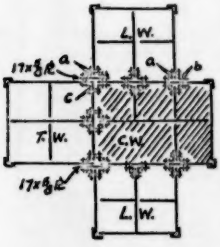
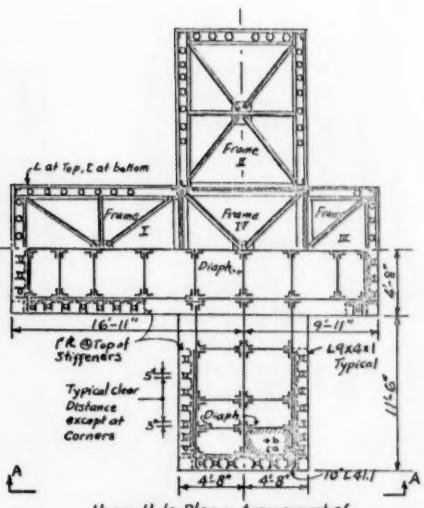
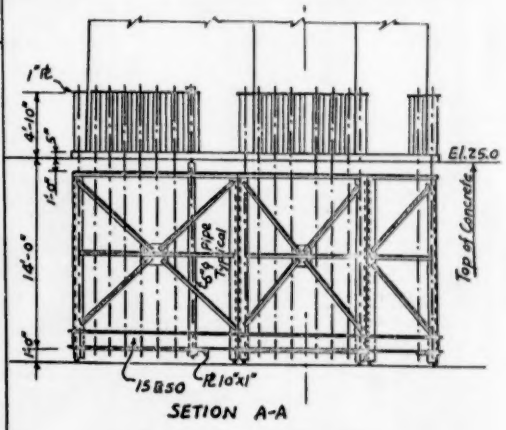


FIG. 9. CONNECTION BETWEEN WINGS

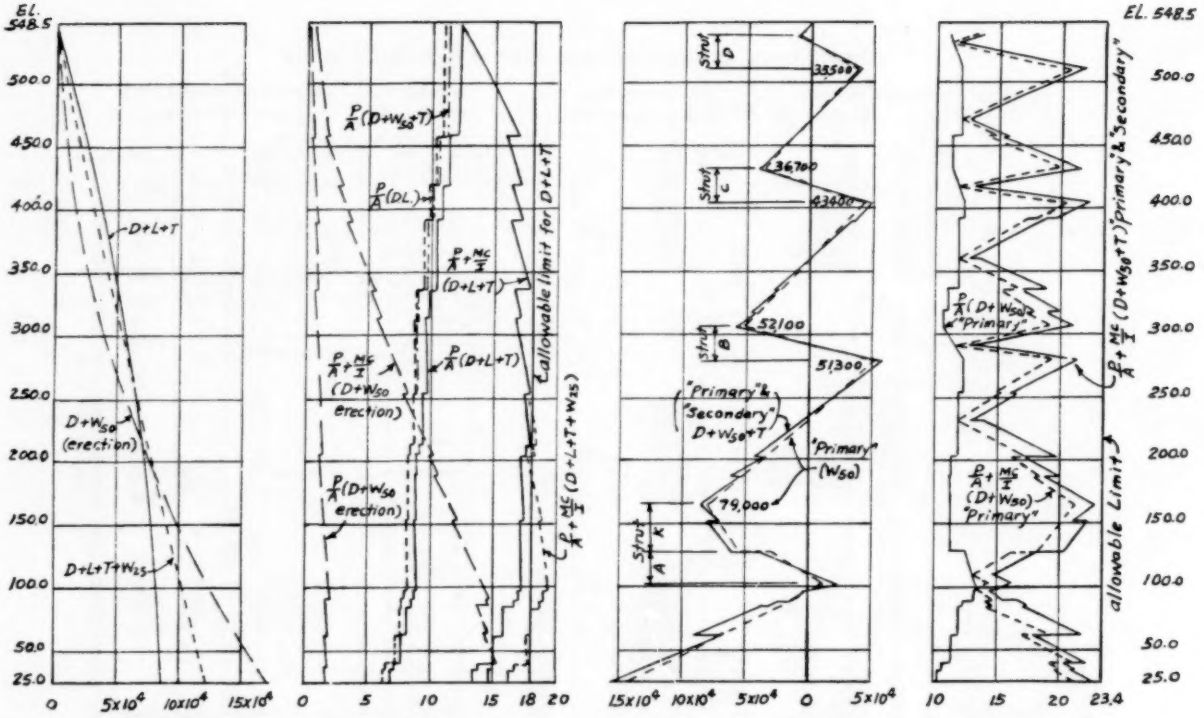


Upper Half Plan – Arrangement of Bracing in Anchor Frame
Lower Half Plan – Arrangement of Anchor Bolts & Dimension of Base R



SETION A-A

FIG 8 TOWER ANCHORAGE



(a) Longitudinal Moments (ft. K.)

(b) Stresses Due to Longitudinal Forces (KSi.)

(c) Transverse Moments (ft. k.)

(d) Stresses Due to Transverse Forces (KSi.)

FIG. 6. STRESS SHEET

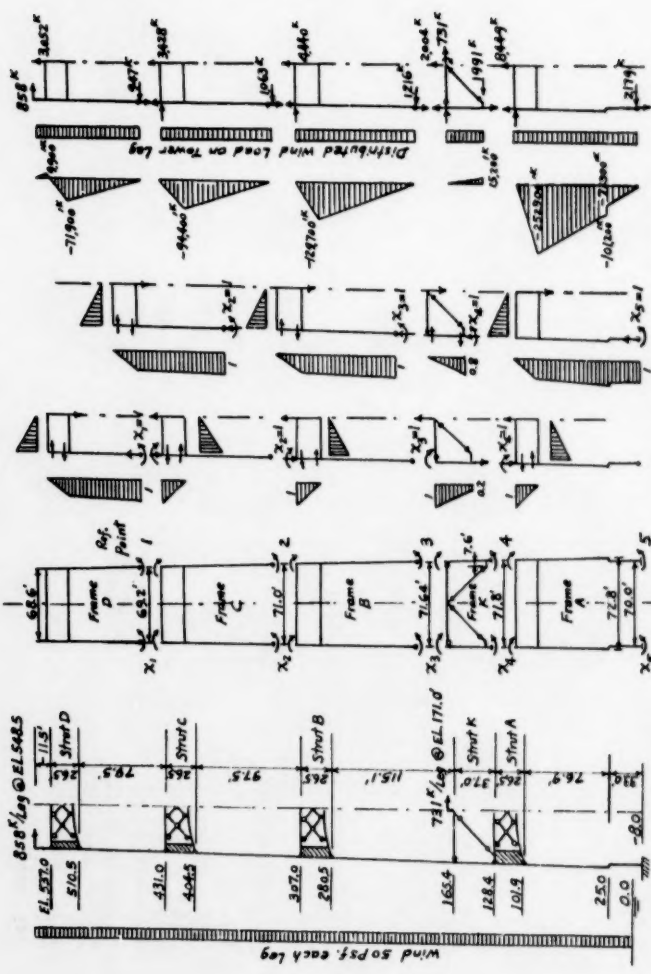


FIG 7. STRESS ANALYSES FOR TRANSVERSE WIND LOAD

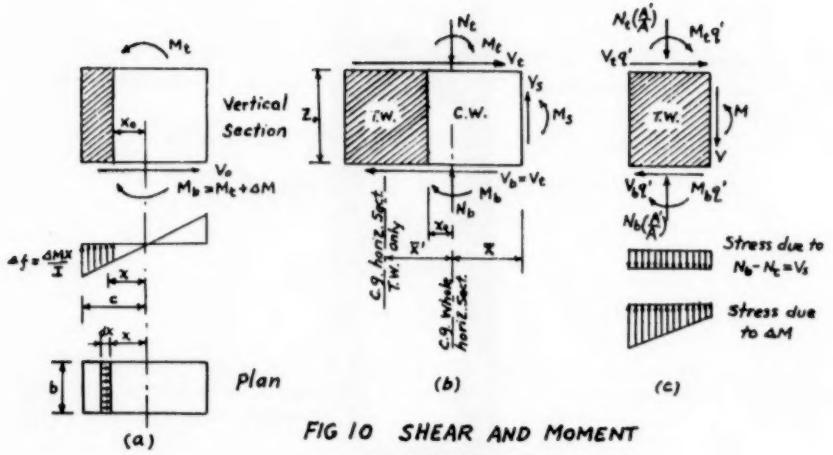
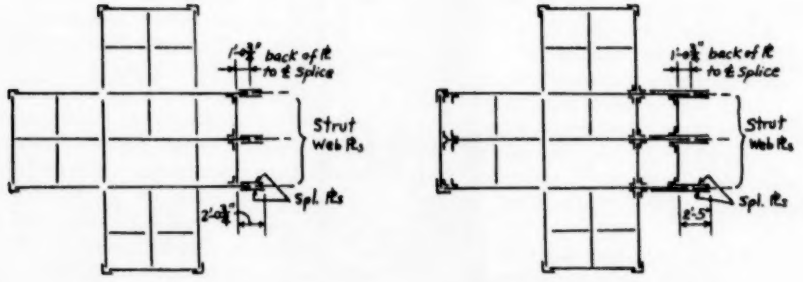


FIG 10 SHEAR AND MOMENT BETWEEN T.W. AND C.W.



(a) Splice Pls for all Struts except at top chords of Struts B & C

(b) Splice Pls at top chords of Struts B and C

FIG 11 SPLICE PLATES FOR STRUTS

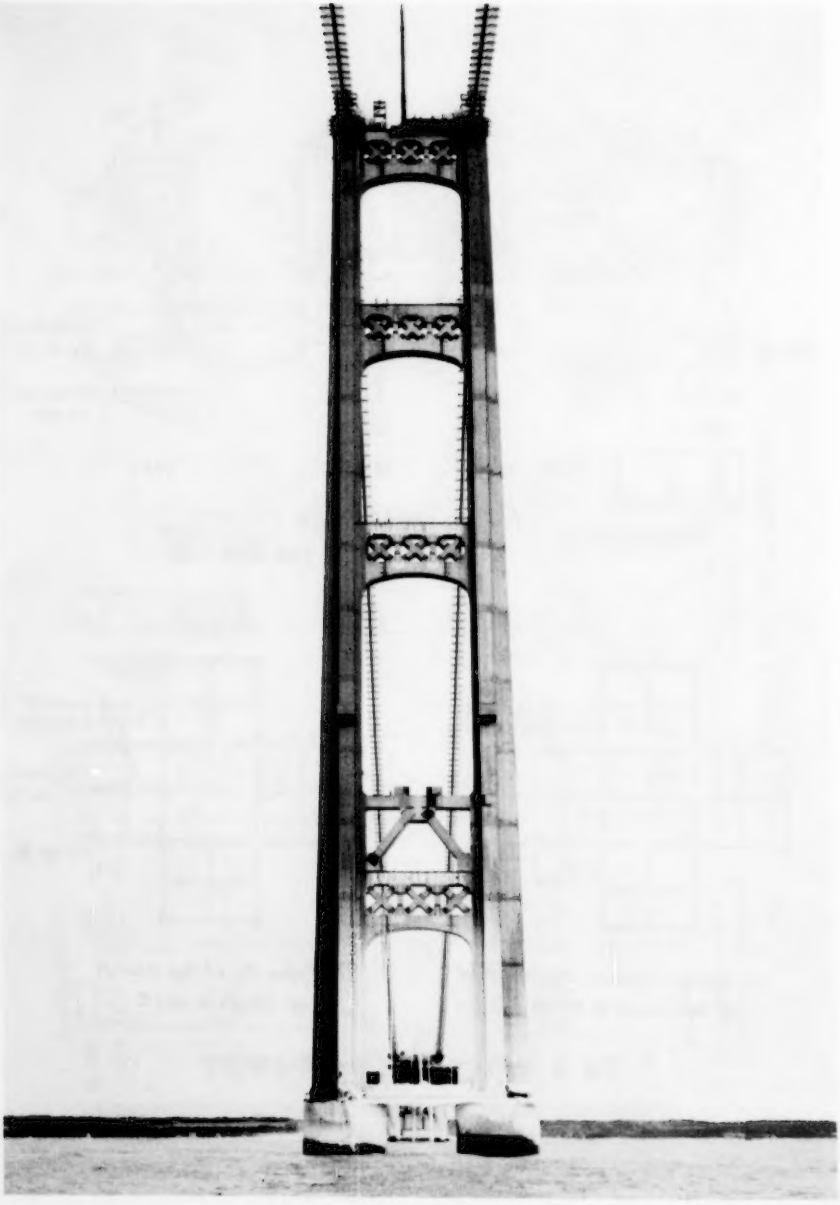


Fig. 12. Spray Ice Coats Base of N. Tower to an Elevation of Nearly 100'.

Paper 1566 March, 1958 ST 2

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STRUCTURAL DIVISION
Proceedings of the American Society of Civil Engineers

AN ELASTIC CRITERION FOR PLASTIC DESIGN

Herbert A. Sawyer, Jr.,[1] M. ASCE
(Proc. Paper 1566)

SYNOPSIS

The significance of the elastic limit to deflection, local instability, and plastic fatigue in plastic design is investigated, and an elastic limit criterion for plastic design is proposed.

INTRODUCTION

Under present specifications, steel building frames and beams are analyzed by the elastic theory and designed so that whether statically determinate or indeterminate they will exceed the elastic limit at the same overload. That is, λ_e, the load factor based on the elastic limit, is uniform for both simple and redundant structures. However, as overloading continues beyond the elastic limit, a favorable redistribution of moments provides an indeterminate structure with a reserve of strength with respect to collapse which is not available to a structure with moments fixed by statics. Thus, for a redundant structure, λ_c, the load factor based on collapse, is usually considerably greater than that for a simple structure.

If collapse strength is the important design criterion, this inequality in collapse load factor is inconsistent and leads to inefficient use of steel. To avoid this inconsistency, both rigid-plastic and elasti-plastic analytical methods for evaluation of collapse load have been presented. The first, commonly termed the "plastic" method, is now highly developed and has been recently summarized by Neal.(1) This method determines the load at which the structure must collapse as a mechanism from formation of plastic hinges; it neglects deformations, and, therefore, any deformational limitations imposed by buckling or rupture. Elasti-plastic methods(2,3,4) which account for

Note: Discussion open until August 1, 1958. A postponement of this closing date can be obtained by writing to the ASCE Manager of Technical Publications. Paper 1566 is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 84, No. ST 2, March, 1958.

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deformations are at once more complex, more powerful, and less well known.

With the use of the plastic method for evaluation of the mechanism-collapse load, it is now possible for the designer to maintain the same mechanism-collapse load factor, λ_{mc}, for redundant structures as has proved completely satisfactory for simple structures, with a resulting considerable saving in material. This uniformity of λ_{mc} is justified and logical, provided, of course, that mechanism-collapse from a single overload is the governing design criterion. Therefore, there is general agreement that plastic methods do not govern if, before a mechanism-collapse occurs, a structure is subject to failure from ordinary fatigue (over, say, 100,000 load applications), excessive deflection, or buckling-collapse.

Certainly, the use of the mechanism-collapse strength as the design criterion for many structures is correct and overdue; the traditional criterion of calculated elastic stress at working load has been overemphasized in the past.

However, as uniformity of mechanism-collapse load factor for simple and redundant structures is gained, the uniformity of elastic limit load factor (λ_e) is lost, and, as Table I shows, for a plastically designed redundant structure this factor may be as low as half of its value for a statically determinate structure. Stated in another way, for a plastic design using load factors derived[5] from AISC standards, the elastic limit may be exceeded at a load as low as 0.83 of the allowable or working load, or at a load including wind or earthquake as low as 0.62 of the allowable load.

Can these situations, and the value of λ_e be neglected in plastic design? Or, should a minimum value be specified for this load factor?

The purpose of this paper is to consider these questions and assess the importance of the elastic limit to a plastic design.

TABLE 1

Load Factors for Design

	Loading	Statically Determinate AISC	Statically Indeterminate Elastic AISC	Plastic Ref.5	Recommended Plastic
Mechanism-collapse Load Factor, λ_{mc}	With wind & earthquake	1.41	1.41 min.	1.41	1.41
	all others	1.88	1.88 min.	1.88	1.88
Elastic Limit Load Factor, λ_e	with wind & earthquake	1.24	1.24	0.62 min*	1.00
	all others	1.65	1.65	0.83 min*	1.27

* May be less for multi-span or variable cross-section structures

Deflection and Permanent Set

Fig. 1 shows relative deflections for elastic and plastic designs for extreme values of the elastic limit load factor, Fig. 1(a) for a structure and loading with a low value, and Fig. 1(b) for a structure and loading with a high value. For the low-value structure the elastic limit for the plastic design is exceeded and the first plastic hinge forms before the allowable load is reached, and the structure must undergo considerable plastic deflection (and the first hinges considerable rotation) before the collapse load is reached. For the high value structure the elastic limit for the elastic and the plastic designs are reached almost simultaneously; very little plastic deformation is required of the plastic design, and the two designs are almost identical.

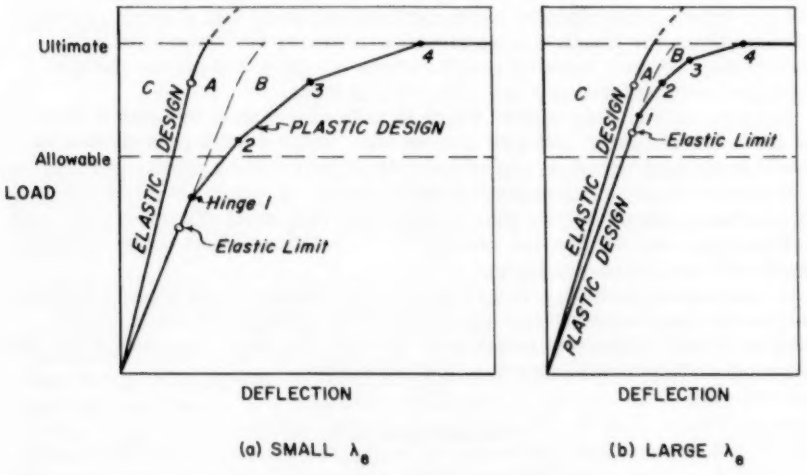

FIG. 1 EFFECT OF ELASTIC LIMIT LOAD FACTOR ON DEFLECTION

The difference in deflection of the elastically and plastically designed structures of Fig. 1 may, for convenience, be considered as the deflection components represented by areas A and B. The relationship between these components and the elastic limit load factor is interesting in that each component tends to decrease as λ_e increases. Area A, which roughly indicates the economy of the plastic design with respect to the elastic design, will clearly approach zero as the two designs approach identity, as will area B, which represents the plastic component of deflection. However, it can also be shown that area C, which represents the elastic deflection of an elastically designed structure, tends to increase as λ_e decreases. Thus, although λ_e is an inverse measure of the additional deflection for plastic design, it is not useful in indicating magnitude of total deflection.

In regard to permanent set, if a structure has no stresses before loading, certain portions of its deflections from a loading become permanent as the maximum stress from the loading exceeds the elastic limit. Actually, every

redundant structure can have initial stresses which can be very high from rolling, fabrication, erection, temperature, and settlement. Thus, in practice, the initial application of even small loadings will cause some unavoidable permanent set. Yet, for subsequent applications of the same load, no further permanent set will be induced, regardless of the magnitude of the load or the initial stresses (unless the geometry of the structure is very unusual). Therefore, for any fixed pattern of loading, the elastic limit is of limited significance in determining the onset or magnitude of permanent set and the deflections resulting therefrom.

For combination or moving loads, the deflection problem becomes more complex, since a deflection failure may occur from the continuing increments of permanent set accompanying either moving loads or the cyclical application of multiple loads. (Of course this phenomenon will not occur if the elastic limit is not exceeded.) However, Horne[6] has shown by a probability analysis that this effect is so improbable for combination loads that it is not of practical importance for plastic design involving ordinary floor and wind loads. Also, neither present building specifications nor proposed plastic design methods are considered applicable to moving loads.

For a certain loading pattern which is fully reversable, the elastic limit has more significance. For this loading type, which a wind-plus-dead-load often closely approximates, the alternating strains will be plastic and continue to be plastic for all loads above the elastic limit. However, such successive increments of permanent set will be approximately equal and opposite in sign, so deflections will not become excessive. (The fatigue effect of these plastic strains will be considered later.)

In conclusion, although a low value of λ_e indicates relatively large plastic-design deflection components and permanent set, permanent set is not inherently objectionable, and assignment of a specific minimum value to λ_e will not necessarily prevent excessive total deflection.

Plastic Instability

A previous investigation[3] has shown that limitations in the curvature capacity of the steel members of a frame may prevent attainment of the ultimate load as given by a plastic analysis. That is, the plastic rotation capacity of one of the first plastic "hinges" to form may be exceeded before all the remaining plastic hinges of the collapse mechanism have formed. Also, it was shown that the rotation capacity of a hinge is governed by (a) statics, and (b) the moment-curvature relationship of the member, and an elasti-plastic method of calculating the ultimate load as governed by these rotation capacities was presented. For steel rolled sections, the ultimate moment and corresponding curvature of the moment-curvature relationship are, in turn, limited primarily by buckling. Therefore, an elasti-plastic analysis based on such a relationship determines the ultimate load as governed by buckling.

As before stated, for a plastically designed structure λ_e, the elastic limit load factor, is an indicator of the relative amount of rotation required of the initial plastic hinges; for high values of λ_e all hinges form almost simultaneously, and little rotation is required of any hinge, whereas for the smaller values of λ_e the initial hinges form at correspondingly smaller fractions of the collapse load, indicating larger required rotations. The possibility then arises that the specification of a proper minimum allowable value for λ_e,

used in conjunction with conventional plastic design, will compensate for the inherent neglect of deformation in plastic design, and account for the failure of some structures to attain the mechanism collapse load because of buckling.

For the investigation of this possibility, the structure and loading shown in Fig. 2 will be used, not only because of its simplicity, but because, for plastic designs of this structure, a, the ratio of span to height, may be varied to give a complete range of values of λ_e. The curves of Fig. 2 for load versus a illustrate this variation. Since P_{mc}, the mechanism-collapse load, is equal to $4M_o/L$, where M_o is the plastic moment, it is constant with respect to a, and in this figure P_{mc} is represented as the uppermost horizontal line. Then, for plastic designs without wind or earthquake, the allowable load would be $P_{mc}/1.88$, as shown by the lowest horizontal line, and for plastic designs including wind or earthquake, the allowable load would be $P_{mc}/1.41$, as shown by the intermediate horizontal line. Assuming a shape factor (ratio of fully plastic moment to moment at elastic limit) of 1.14, the curve shown for P_e, the elastic limit load, may be calculated by the elastic theory:

$$P_e = \frac{P_{mc}}{1.14}\left(\frac{a+6}{2a+6}\right) \qquad (1)$$

Dividing this load by the allowable loads to obtain λ_e, it is seen that as a approaches zero, λ_e approaches its maximum values of 1.65 and 1.24, and that as a approaches infinity as a limit, λ_e approaches its minimum values of 0.83 and 0.62 (Table I).

Obviously, accuracy of the calculated elasti-plastic or buckling-limited strength of this frame depends primarily on the validity of the moment-curvature (M-ϕ) relationships assumed. Cross-sectional proportions, moment-gradient or shear, lateral support and other factors influence these relationships in probably significant, but, with present knowledge, somewhat indeterminate degree. Fig. 3 shows the moment-curvature relationship assumed for this frame. This relationship is conservatively similar to determinations by Beedle[7] for 8WF40 members and a relationship suggested by Ketter.[8] Also, it agrees with the statement by Thurlimann[9] that local buckling is not usually encountered in WF-beams before the strain in the steel reaches a value of 0.0014, which corresponds to a value of ϕ/ϕ_e of about 12. Also, Beedle[10] has observed that at a stage corresponding to region A, Fig. 3, local wrinkling of the compression flange eventually occurs causing subsequent reduction of moment.

Ultimate loads of the frame as given by elasti-plastic calculations and the assumed M-ϕ relationship are shown by the heavy curve of Fig. 2. Note that, for the assumed conditions, buckling prevents attainment of the mechanism-collapse load for frames with span-to-height ratio of over 2, that is, for such frames λ_c becomes significantly less than the values of 1.41 or 1.88 maintained for λ_{mc}. This trend continues until λ_c becomes less than 1.00 for wind-load plastic designs with a greater than 10. Of course, this condition is intolerable to the designer, and the allowable loads must be so modified that the value of λ_c is essentially maintained for all values of a.

If plastic instead of elasti-plastic methods are used in design it is proposed that the desired value of λ_c be essentially maintained by assigning minimum values to λ_e. This minimum value could be 1.00 for loadings including wind or earthquake, and 1.27 for all other loadings. Although the curves

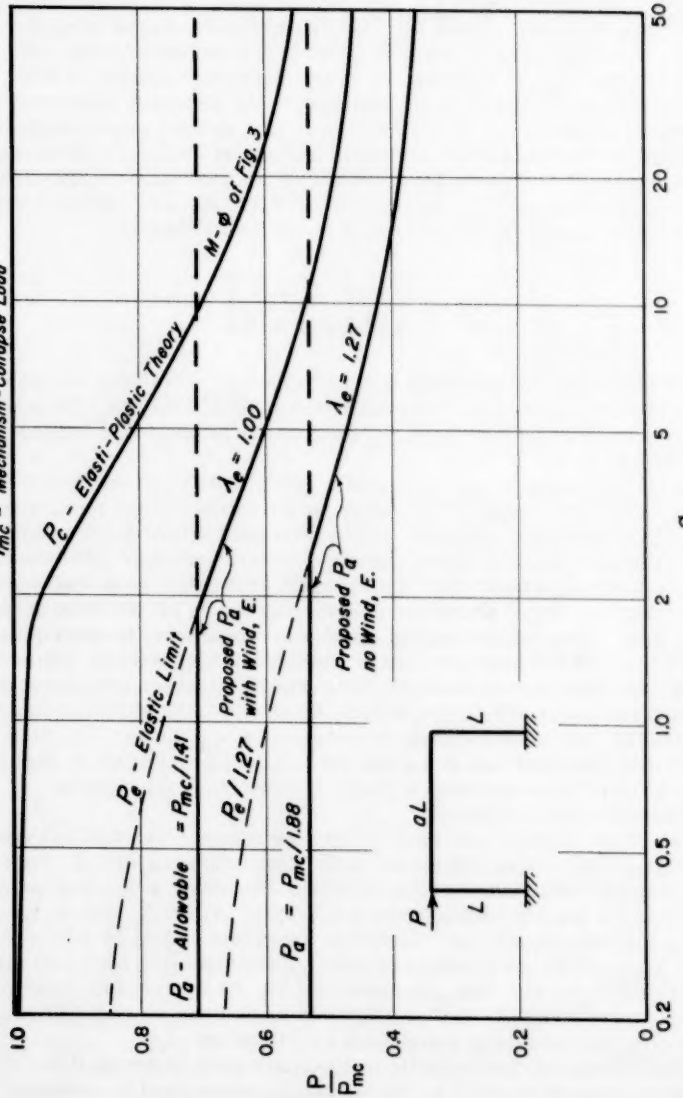

FIG. 2 PROPOSED ELASTIC LIMIT CRITERION FOR PLASTIC DESIGN

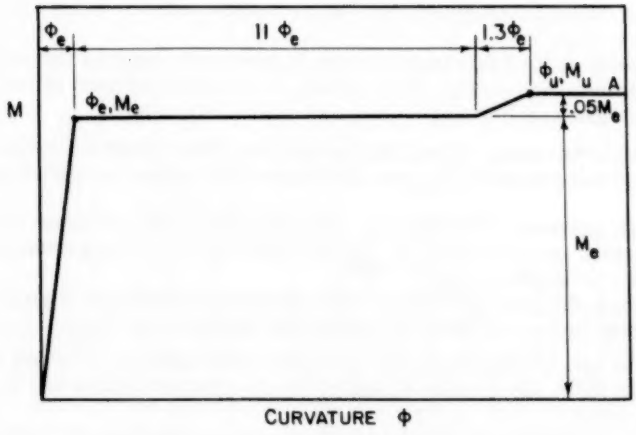

FIG. 3 ASSUMED MOMENT-CURVATURE RELATIONSHIP

of Fig. 2 for these values of λ_e demonstrate that they will maintain an adequate λ_c for the assumed conditions, future investigations may indicate modifications of these assumptions and the values based on them. The important conclusion, which is largely independent of the quantitative accuracy of the above assumptions, is that some minimum value of λ_e should be observed in plastic design.

In design practice, the adoption of this proposal would mean that the elastic stresses in a plastically designed structure could not exceed 33,000 psi for allowable loads which include wind or earthquake, or 26,000 psi for other allowable loads.

Plastic Fatigue

The possibility must be considered that the alternating plastic strains previously discussed in relation to deflections may eventually cause a fracture at less than the static collapse load, such a failure possibly being similar to a fatigue fracture.

Assessment of the importance of such a possibility involves the correlation of bodies of information from three areas, or the establishment of three continuous functions:

(a) Test of Materials. The failure-producing range of strain in the structural material, $\Delta\epsilon$, as a function of the number of load-reversals, n.

(b) Strain Analysis of Structures. The maximum range of strain in a structure per reversal, $\Delta\epsilon_{max}$, as a function of the magnitude of load on the structure P/P_{mc}.

(c) Loading-Frequency Analysis. The number of reversals of load, n, as a function of the magnitude of load on the structure, P/P_{mc}.

Plastic fatigue is important only for reversable loadings, of which wind loadings are the most common example. Hence this discussion will be limited to wind loadings.

The essential principles for determination of the function for n vs. P/P_{mc}, function (c) above, have been outlined by Horne[6] and Freudenthal.[17] Horne also found that, for a certain wind-climate, a load magnitude which can be expected to have 10 reversals will be only 0.53 of the maximum expected single load, and thus, if the elastic limit is not exceeded at 0.53 of the collapse load, alternating plasticity involving 10 reversals or more cannot be expected to occur. However, from Table I, the elastic limit load for some structures may be less than 0.53 and as low as 0.44 (0.62/1.41) of the mechanism-collapse load. For such structures the probability of alternating plasticity will be significant, according to Horne's standard. Its relative importance will now be assessed by the establishment and correlation of the three functions listed above.

The following analysis includes, of necessity, many approximations which are rougher than those customary to engineering computations. However, these approximations would be unjustified only if their probable inaccuracies, in aggregate, could alter the conclusions which will be drawn. In the author's opinion they could not.

Plastic Fatigue Strength of Material

Little work had been done in this area until the Pressure Vessel Research Committee of the Welding Research Council (WRC) initiated and coordinated an extensive research program, the results of which have been summarized and interpreted by Kooistra.[11] Following is an abstract of these conclusions as they apply to the problem at hand.

Although for conventional fatigue, involving more than 100,000 cycles, fatigue strength is related to the tensile properties of a steel, resistance to fatigue failure from less than 10,000 cycles (20,000 reversals) is largely dependent on the strain range and independent of type of steel, as shown by the WRC tests on steels ranging from A201 with 36,000 psi yield strength to T-1 with 123,000 psi yield strength. Because of this fact, and because of the similarity of A201 pressure-vessel steel to A7 structural steel, it is assumed that the following results of the WRC program for A201 steel apply to A7 steel:

Reversals n	Strain Range—per cent for fracture $\Delta\epsilon$
100,000	0.34
10,000	1.1
1,000	(4.) extrapolated

For the purpose of the present investigation of wind loaded structures, the author has very arbitrarily assumed the following additional values:

100	13.
10	38.
1	100.

All the above strain ranges are actual strain ranges at the point of subsequent fracture. Since slight imperfections cause strain concentrations, the above experimental determinations (for 100,000 and 10,000 cycles) were, of necessity, obtained from machined and ground specimens. Tests of specimens with imperfections and discontinuities of course resulted in failures at lower values of nominal strain-range. Strain concentration factors were then determined as the ratio of strain-range for failure of a ground specimen to nominal strain-range for failure of an imperfect specimen, the number of reversals being equal. By this means the following strain concentration factors were obtained in the WRC program:

Ground surface	1.0
Rolled plate surface	1.6
0.01-in.-deep notch	2.7
0.06-in.-deep notch	3.8
Welded connection, nozzle to pressure vessel	3.9 to 5.0

Although these factors undoubtedly change appreciably as the increasing strains at an imperfection exceed the elastic range at point after point, information on this variation was not presented in the report. For this analysis, these factors will be assumed constant for all combinations of elasticity and plasticity.

Strain Analysis of Structures

In Fig. 4 the strain range at the maximum concentration of strain in various

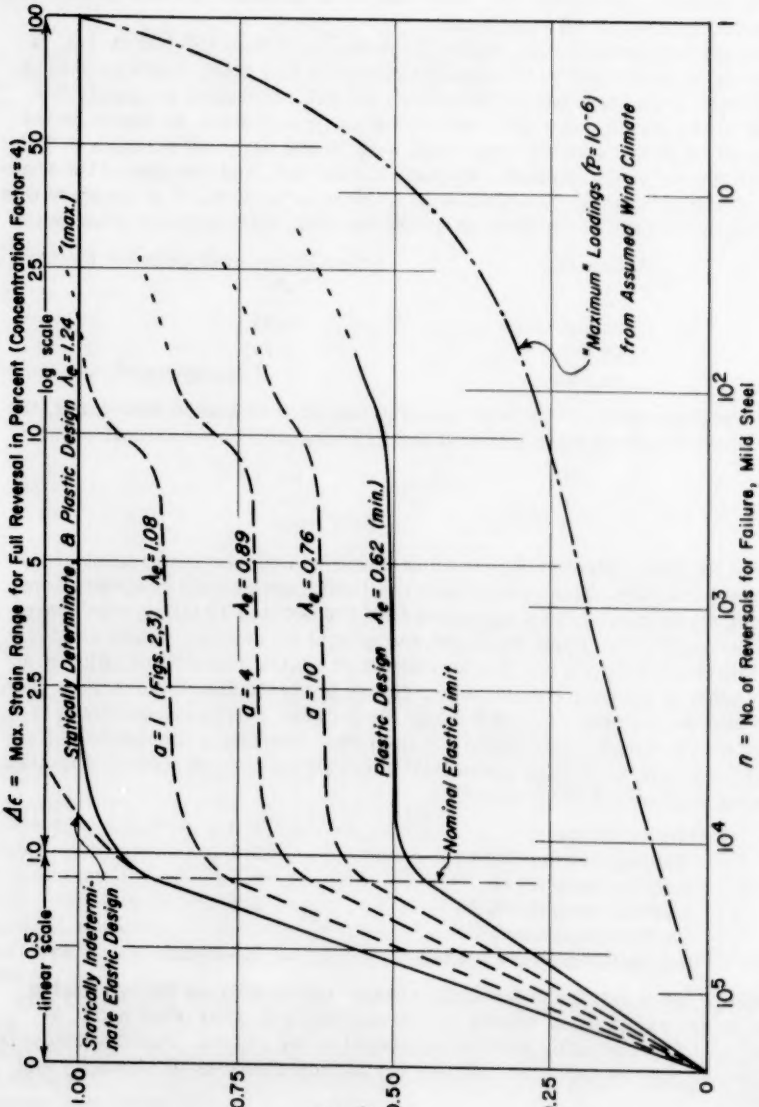

FIG. 4 PLASTIC FATIGUE AND PLASTIC DESIGN FOR WIND

plastically designed structures of constant cross-section for full load-reversal is plotted versus the relative load, P/P_{mc}. The strain concentrations plotted have been obtained by multiplying the nominal strains by an assumed strain concentration factor of 4, which would seem reasonable since plastic hinges are frequently located at welded corners, connections, or stiffeners. The upper solid-line curve is the upper envelope (neglecting strain-hardening) for plastically designed structures. For structures corresponding to this curve the elastic-limit load factor is maximum at 1.24, and all plastic hinges form simultaneously. This curve coincides with the curve for an elastically designed statically determinate structure. The lower solid curve is the lower envelope for plastically designed structures. This curve represents a structure for which λ_e is minimum at 0.62 and for which the initially formed plastic hinges must undergo excessive deformations before formation of all hinges of the collapse mechanism. By definition, the curve of any plastically designed structure must lie between these envelope curves, and its location will depend on structural dimensions, loading, and moment-curvature relationships. As examples, approximate curves (obtained by elasti-plastic[3] computations) are shown for structures defined by Fig. 2 with values of a of 1, 4, and 10, and having the moment-curvature relationship of Fig. 3. As has been seen, stability considerations limit this relationship and make it probable that the maximum load for each of these structures will be attained at a concentration strain range of about 4 x 2 x 0.014, or 11%. If, for this study of fatigue, instability is neglected, M of the M-ϕ relationship will continue to increase for large increases in strain, and the load-strain curves for these structures would have extensions similar to those shown.

Loading Frequency Analysis

The author is indebted to the previously mentioned work of Horne[6] for several steps in the following analysis which parallel or were inspired by this work.

For this analysis, the wind loading on a structure is defined by assuming the following conditions, which, it is believed, produce a severe, and thus conservative, wind loading:

1. The life of the structure is 200 years.
2. Although the wind load on a structure is actually a vector quantity which varies continuously with time, for the present purpose of obtaining reversals, a succession of discrete loads will be assumed applied to the structure, each consisting of the maximum wind load actually experienced by the structure between successive major changes in direction of the wind.
3. In the life of the structure, four loadings of between 57.7 and 58.5 miles per hour, and 80 loadings of between 42.5 and 43.5 mph will occur. For comparative purposes these data have been set to agree roughly with the loading data used by Horne, based on wind velocities observed at the English airship works at Cardington. These frequencies are roughly equivalent to frequencies observed by the United States Weather Bureau at Dayton, Ohio, less severe than those at Cleveland, and more severe than those at Cincinnati.[12]
4. Half of these loadings will be from a north wind and half from a south wind, north and south being the critical loading directions for the structure.

5. The north-and-south loading frequency for all other wind speeds would be as given by a normal frequency distribution curve defined by the above frequencies for 58 and 43 mph. (Actually, only the 40-to-80-mph ranges of this curve have real significance in the following analysis.)

It is assumed that a structure of a height corresponding to the usual height of wind observations and subjected to this loading would be designed for a working load of 25 psf, which, multiplied by the collapse load factor of 1.41 from Table I, would yield a collapse load of 35.2 psf. These loads, actually combinations of windward pressure and leeward vacuum, will be further assumed, as has been previously recommended,[13] to be 1.3 times the stagnation pressure from the wind for standard conditions, or

$$p = (1.3)\,0.00256\,V^2 \tag{2}$$

where p is in psf and V is in mph. The working load wind velocity will then be 87 mph and the theoretical collapse load wind velocity 103 mph.

With loading pattern and working load defined, statistical-mathematical methods will now be used to determine, first, the probability per structure of a single application of the collapse load, and, second, the lower level of load for which there will be a similar probability of n reversals.

The equation of the normal frequency distribution curve is of the form:

$$y = \frac{h}{\sqrt{\pi}}\, e^{-h^2V^2}$$

where y is a measure of the frequency of a wind of velocity V, and h is a constant. Since the total area under this curve is unity, any differential area $y\Delta V$ is the probability of any measured wind lying in the interval ΔV. From loading assumptions 3 and 4 this probability for a north wind of from 57.5 to 58.5 mph is 2/N, where N is the total number of winds or wind reversals. Hence

$$\frac{2}{N} = y\Delta V = \left(\frac{h}{\sqrt{\pi}}\, e^{-h^2V^2}\right)1$$

Writing a similar equation for the 42.5 to 43.5 mph interval and solving simultaneously, N is found to be 63,000 reversals per 200 years, and the frequency distribution curve is found to be

$$y = 0.0251\, e^{-0.00198V^2} \tag{3}$$

which is shown in Fig. 5 for the critical range for positive (north) speeds.

The probability of any single wind exceeding a velocity V_1 is the area under the curve to the right of V_1, or

$$p = 0.0251\int_{V_1}^{\infty} e^{-0.00198V^2}dV = 0.563\int_{t_1}^{\infty} e^{-t^2}dt$$

where $t = 0.0446V$.

Evaluating by integrating successively by parts, the following asymptotic

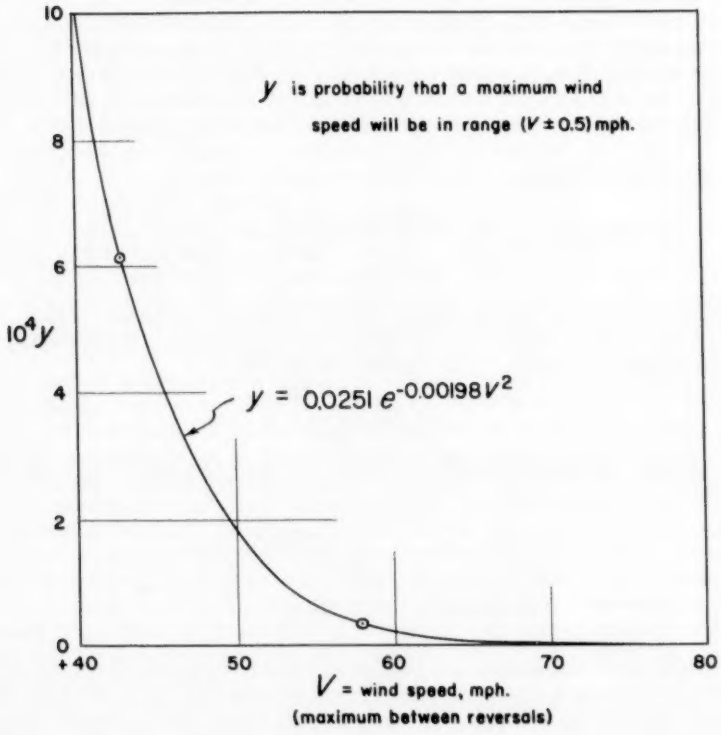

FIG. 5 SIGNIFICANT PORTION OF FREQUENCY DISTRIBUTION CURVE OF MAXIMUM WINDS

series is obtained:(14)

$$p = \frac{e^{-t_1^2}}{3.54\,t_1}\left(1 - \frac{1}{2t_1^2} + \frac{1\cdot 3}{(2t_1^2)^2} - \frac{1\cdot 3\cdot 5}{(2t_1^2)^3}\cdots\right) \qquad (4)$$

The probability of either a velocity exceeding V_1 or a velocity less than $-V_1$ in the life of the structure is $2Np$. This probability for the theoretical collapse wind of 103 mph is 5.35×10^{-6}, which is the probability of failure for a structure during a 200-year life from a single high wind.

For more than one application of a high wind, the probability that one high wind exceeding V_1 in absolute magnitude is the reverse of the previous high wind exceeding V_1 in absolute magnitude is practically 0.5. Hence, the probability in the life of a structure of a wind which: (1) exceeds V_1 or is less than $-V_1$ and, (2) is a reversal of the previous wind of like magnitude, is:

$$P = 0.5(2Np)$$

$$= 63{,}000\,p = \frac{18{,}000\,e^{-t_1^2}}{t_1}\left(1 - \frac{1}{2t_1^2} + \frac{1\cdot 3}{(2t_1^2)^2}\cdots\right) \qquad (5)$$

The probability of obtaining n such load reversals in the life of the structure may be obtained approximately from the Poisson Distribution since N is large and p is small. This probability is:(15)

$$P_n = \frac{e^{-P}P^n}{n!}$$

It follows that the probability of obtaining n or more such load reversals will be:

$$P_{\substack{n=n \text{ to} \\ n=63{,}600}} = e^{-P}\sum_{n=n}^{n=63{,}600}\frac{P^n}{n!} \qquad (6)$$

It would seem reasonable and conservative that an allowable value of this probability of failure from n load reversals be less than the previously determined probability for failure from one load. Hence the allowable value of the probability of Eq. (5) will be assumed to be 10^{-6} (use of 10^{-5} or 10^{-7} changes result less than 2%).

Now, using any desired value of n, Eqs. (5) and (6) may be solved by trial-and-error for a corresponding value of V_1, the minimum wind velocity of the n reversals.

However, the average load for the n reversals, rather than the minimum load, is desired. Since, from Eq. (2), the load is proportional to the square of V, and assuming the n winds will have the velocity distribution of Eq. (3), the corresponding value of V will be

$$\bar{V} = \sqrt{\frac{\int_{V_1}^{\infty} V^2 y\,dV}{\int_{V_1}^{\infty} y\,dV}} = \frac{1}{0.0446}\sqrt{\frac{1}{2} + \frac{t_1^2}{1 - \frac{1}{2t_1^2} + \frac{1\cdot 3}{(2t_1^2)^2}\cdots}} \qquad (7)$$

The results of the computations necessary for this analysis follow:

n	V_1^*	$\bar{\bar{V}}$	$P/P_{mc} = (\bar{\bar{V}}/103)^2$
10	66.5	70	.462
100	50	54.5	.278
1000	41	45.5	.196
63,000	0	16	.024

The importance of alternating plasticity as a source of failure is determined by the graphic combination shown in Fig. 4 of the above three bodies of data. The plotting of data from (b) for strain range versus the load has been discussed previously. The data of (a) for number of reversals versus strain range is used to lay-off an n scale parallel to the strain range scale: that is, each value of n is placed at the corresponding value of strain range required for failure. Finally, the values of the relative applied load, or P/P_{mc}, from (c) are plotted against the corresponding values of n, resulting in the lower curve of the figure.

It is clear that the relative vertical position of the general n-versus-load curve and the particular strain-range-versus-load curve for a structure indicates the seriousness of fatigue failure throughout the range of possible values of n for the structure. Where the former is below the latter curve, the probability of fatigue failure is so much smaller than the probability of failure from a single overload that it has no design significance. Where the former is above the latter curve, the probability of fatigue failure becomes governing, and the design must be modified. Since the value of the elastic limit load factor is the best single indicator of vertical position of the latter curve, this modification could consist of assigning a minimum value to this load factor.

Since in Fig. 4 the n-versus-load curve is considerably below all possible strain-versus-load curves, it may be concluded that fatigue is never critical for plastically designed structures loaded with the assumed wind loads. Also, since the computations for this analysis demonstrated to the author that the position of the n-versus-load curve is little altered by wide variations in analytical assumptions, and since the assumptions made were generally on the safe side, the author feels justified in concluding that fatigue is never critical for the wind loading of plastically designed structures which have been adequately designed for single overloads.

Lazard[16] has reported plastic fatigue tests conducted in France on rolled beams. Beams with drilled holes in flanges subjected to about 16 complete stress reversals above the elastic limit did not rupture. However, a beam with punched holes ruptured suddenly and with brittleness after only six such reversals at a moment 25% above the elastic limit moment and a deflection 50% above the elastic limit deflection. In the author's opinion this test is significant in showing that repeated reversals increase the tendency for sudden brittle rupture as well as for the more usual ductile rupture or progressive fatigue rupture. Also, these tests again show that brittle ruptures violate the established criteria for the usual types of rupture—no reasonable assumption for a strain concentration factor could reconcile this test to the WRC strain ranges. Therefore, the usual preventive measures for brittle

*The error in these values from assumption of a Poisson Distribution is always conservative and increases with n to about 10% for $n = 1000$.

fracture have additional importance for plastic design, including detailing which minimizes strain concentrations (punched holes must be properly reamed) and the use of brittle-fracture-resistant steel (Thomas steel, which has no chemical-analysis specifications, was used for this test).

CONCLUSIONS

1. Although the elastic limit load factor, λ_e, has qualitative significance for deflection in plastic design, specification of a minimum value of λ_e is not a satisfactory quantitative control of such deflection.
2. Specification of a proper minimum value of λ_e will prevent premature failure of a structure from local buckling prior to formation of the collapse mechanism. Suggested minimum values of λ_e are 1.00 for loads including wind and earthquake and 1.26 for all other loads. These minima are equivalent to maximum allowable elastic stresses of 33,000 psi and 26,000 psi, respectively.
3. Although the probability of failure from repeated alternating plastic strains from wind loadings depends on λ_e, the probability of such failure, regardless of the number of repetitions, is too small to influence a plastic design.

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MOMENTS IN BEAMS BY THE METHOD OF PARTIAL MOMENTS

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SYNOPSIS

This paper presents a method of obtaining moments in restrained or continuous beams by first solving for the moments at each support with the loading placed on one span at a time only (PARTIAL MOMENTS) and, then, by combining these moments in a prescribed manner, getting the moments at each support with all the spans loaded (TOTAL MOMENTS). Two solutions are presented—one is analytical, the other is graphical.

INTRODUCTION

During the past two or three decades a good deal of attention has been devoted in engineering literature to the matter of devising methods for obtaining moments in restrained or continuous beams which are simpler and less time consuming than the Theory of Three Moments, Slope Deflection or similar methods. Thus, in 1925 there appeared The Method of Conjugate Points,(1) a graphical solution and in 1930, The Method of Distributing Fixed-End Moments,(2) an analytical solution. The method described in this paper is called by the writer THE METHOD OF PARTIAL MOMENTS. The solution of the Partial Moments may be carried out either analytically or graphically as will be shown.

The advantages claimed for this method are as follows: it is less time-consuming than the older methods, particularly when more than four spans are involved and when several trial loadings are to be investigated. The method is less subject to errors because its application is self-checking and it does not require the use of computing machines because the ordinary use of slide-work in common practice gives results well within the limits of required accuracy in practical work.

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General Description of Method

The continuous beam is divided into as many "units" as there are spans in the beam. Each "unit" includes THREE spans, FOUR points of support, ends free and with loading on one span only (See Plate I, Figs. 2a to 2f). The moments at the supports adjacent to the loaded span only are solved for and are referred to as PARTIAL MOMENTS. The partial moments are then combined as shown on Plate V to give the TOTAL MOMENTS at the supports of the continuous beam.

Basic Principles and Method of Procedure

The underlying principles of the method and the method of procedure in solving illustrative problems will now be given.

Let Fig. 1a on Plate I represent a continuous beam comprising five spans, six points of support with the ends of the beam simply supported. The usual equations of condition of the Theorem of Three Moments may be written as follows:

$$0 + 2(L_1I_2+L_2I_1)M_2 + L_2I_1M_3 = -\tfrac{1}{4}W_1(L_1)^3I_2 - \tfrac{1}{4}W_2(L_2)^3I_1 \quad (1)$$

$$L_2I_3M_2 + 2(L_2I_3+L_3I_2)M_3 + L_3I_2M_4 = -\tfrac{1}{4}W_2(L_2)^3I_3 - \tfrac{1}{4}W_3(L_3)^3I_2 \quad (2)$$

$$L_3I_4M_3 + 2(L_3I_4+L_4I_3)M_4 + L_4I_3M_5 = -\tfrac{1}{4}W_3(L_3)^3I_4 - \tfrac{1}{4}W_4(L_4)^3I_3 \quad (3)$$

$$L_4I_5M_4 + 2(L_4I_5+L_5I_4)M_5 + 0 = -\tfrac{1}{4}W_4(L_4)^3I_5 - \tfrac{1}{4}W_5(L_5)^3I_4 \quad (4)$$

Eqs. (1) to (4) may be solved by either the method of determinants or by the method of simultaneous equations, quite laborious operations. The solution becomes even more involved if the moments are to be expressed in terms of the loadings "w" or "P" on the several spans. The method presented in this paper makes it possible to obtain moments in terms of loadings with comparative ease.

In applying the method of Partial Moments we must bear in mind the interrelation which exists between the loadings on the several spans of a beam continuous over a number of supports and the moments at the supports. This interrelation may be expressed in the form of two propositions:

Proposition I:

The loading on any one span of a continuous beam induces moments at all points of support over which the beam is continuous or at which restraint exists. The signs of these moments are NEGATIVE at the supports adjacent to the loaded span and alternately POSITIVE and NEGATIVE at each preceding and each succeeding point of support.

Proposition II:

The TOTAL MOMENT at each point of support of a continuous beam loaded on several or all spans is equal to the algebraic sum of the moments induced at these supports by the separate loadings on each span.

Thus, referring to Figs. 1b to 1f on Plate I, on the basis of Proposition I,

the moments at the supports will be:

Supports	#2	#3	#4	#5	
For loading on span 1	$-M_2^1$,	$+M_3^1$,	$-M_4^1$,	$+M_5^1$	(5)
For loading on span 2	$-M_2^2$,	$-M_3^2$,	$+M_4^2$,	$-M_5^2$	(6)
For loading on span 3	$+M_2^3$,	$-M_3^3$,	$-M_4^3$,	$+M_5^3$	(7)
For loading on span 4	$-M_2^4$,	$+M_3^4$,	$-M_4^4$,	$-M_5^4$	(8)
For loading on span 5	$+M_2^5$,	$-M_3^5$,	$+M_4^5$,	$-M_5^5$	(9)

Again, on the basis of Proposition II, the TOTAL MOMENT at each support will be:

At support # 2	$M_2 = -M_2^1 - M_2^2 + M_2^3 - M_2^4 + M_2^5$	(10)
At support # 3	$M_3 = +M_3^1 - M_3^2 - M_3^3 + M_3^4 - M_3^5$	(11)
At support # 4	$M_4 = -M_4^1 + M_4^2 - M_4^3 - M_4^4 + M_4^5$	(12)
At support # 5	$M_5 = +M_5^1 - M_5^2 + M_5^3 - M_5^4 - M_5^5$	(13)

In Eqs. (5) to (13) the SUPERSCRIPTS refer to the number of the LOADED SPAN which induces the moments, while the SUBSCRIPTS refer to the number of the SUPPORT at which the moment obtains.

Now, it is not necessary to solve for the moments at all the supports for each loaded span. It is sufficient to solve for the moments at the supports adjacent to the loaded span only and then make the summation in a manner which will be explained later.

Now, let Fig. 2a represent the beam shown in Fig. 1a. Also lets Figs. 2b to 2f take the place of Figs. 1b to 1f respectively, differing from the latter, however, in that they each include three spans, and four points of support instead of five spans and six points of support. The length of the end spans C_1L_2, C_1L_3, C_2L_3 and C_rL_4 must be such as to make the moments

		M_2^1 in	Fig. 2b
M_2^2	and	M_3^2 in	Fig. 2c
M_3^3	and	M_4^3 in	Fig. 2d
M_4^4	and	M_5^4 in	Fig. 2e
and M_5^5		in	Fig. 2f

equal in magnitude to the corresponding moments in Figs. 1b to 1f respectively. These moments will be referred to hereinafter as PARTIAL MOMENTS. The general expressions for the end span factors C_1 and C_r are given on Plate I.

The next step is to solve for the PARTIAL MOMENTS in terms of the loading on the several spans. In the GRAPHICAL solution these moments are obtained directly from charts VI to XVII. In the ANALYTICAL solution this is done by writing the usual Theorem of Three Moments equations as follows:

For Uniform Loading:

For Fig. 2b

$$0 + 2(L_1 I_2 + L_2 I_1) M_2' + L_2 I_1 M_3' = -\tfrac{1}{4} W_1 (L_1)^3 I_2 - 0 \quad (14)$$

$$L_2 I_3 M_2' + 2(L_2 I_3 + C_R L_3 I_2) M_3' + 0 = - 0 - 0 \quad (15)$$

Solve for M_2^1

For Fig. 2c

$$0 + 2(L_1 I_2 + L_2 I_1) M_2^2 + L_2 I_1 M_3^2 = - 0 - \tfrac{1}{4} W_2 (L_2)^3 I_1 \quad (16)$$

$$L_2 I_3 M_2^2 + 2(L_2 I_3 + C_R L_3 I_2) M_3^2 + 0 = -\tfrac{1}{4} W_2 (L_2)^3 I_3 - 0 \quad (17)$$

Solve for M_2^2 and M_3^2

For Fig. 2d

$$0 \quad 2(C_L L_2 I_3 + L_3 I_2) M_3^3 + L_3 I_2 M_4^3 = - 0 - \tfrac{1}{4} W_3 (L_3)^3 I_2 \quad (18)$$

$$L_3 I_4 M_3^3 + 2(L_3 I_4 + C_R L_4 I_3) M_4^3 + 0 = - \tfrac{1}{4} W_3 (L_3)^3 I_4 - 0 \quad (19)$$

Solve for M_3^3 and M_4^3

For Fig. 2e

$$0 + 2(C_L L_3 I_4 + L_4 I_3) M_4^4 + L_4 I_3 M_5^4 = - 0 - \tfrac{1}{4} W_4 (L_4)^3 I_3 \quad (20)$$

$$L_4 I_5 M_4^4 + 2(L_4 I_5 + L_5 I_4) M_5^4 + 0 = - \tfrac{1}{4} W_4 (L_4)^3 I_5 - 0 \quad (21)$$

Solve for M_4^4 and M_5^4

For Fig. 2f

$$0 + 2(C_L L_3 I_4 + L_4 I_3) M_5^4 + L_4 I_3 M_5^5 = - 0 - 0 \quad (22)$$

$$L_4 I_5 M_5^4 + 2(L_4 I_5 + L_5 I_4) M_5^5 + 0 = - 0 - \tfrac{1}{4} W_5 (L_5)^3 I_4 \quad (23)$$

Solve for M_5^5

For Concentrated Loading

For Fig. 3b

$$0 + 2(L_1 I_2 + L_2 I_1) M_2' + L_2 I_1 M_3' = - P_1' (L_1)^2 I_2 K_A' - P_1'' (L_1)^2 I_2 K_A'' - 0 \quad (14a)$$

$$L_2 I_3 M_2' + 2(L_2 I_3 + C_R L_3 I_2) M_3' + 0 = - 0 - 0 - 0 \quad (15a)$$

Solve for M_2^1

For Fig. 3c

$$0 + 2(L_1 I_2 + L_2 I_1)M_2^2 + L_2 I_1 M_3^2 = -0 - P_2'(L_2)^2 I_1 K_B' - P_2''(L_2)^2 I_1 K_B'' \quad (16a)$$

$$L_2 I_3 M_2^2 + 2(L_2 I_3 + C_R L_3 I_2)M_3^2 + 0 = -P_2'(L_2)^2 I_3 K_A' - P_2''(L_2)^2 I_3 K_A'' - 0 \quad (17a)$$

Solve for M_2^2 and M_3^2

For Fig. 3d

$$0 + 2(C_L L_2 I_3 + L_3 I_2)M_3^3 + L_3 I_2 M_4^3 = -0 - P_3'(L_3)^2 I_2 K_B' - P_3''(L_3)^2 I_2 K_B'' \quad (18a)$$

$$L_3 I_4 M_3^3 + 2(L_3 I_4 + C_R L_4 I_3)M_4^3 + 0 = -P_3'(L_3)^2 I_4 K_A' - P_3''(L_3)^2 I_4 K_A'' - 0 \quad (19a)$$

Solve for M_3^3 and M_4^3

For Fig. 3e

$$0 + 2(C_L L_3 I_1 + L_4 I_3)M_4^4 + L_4 I_3 M_5^4 = -0 - P_4'(L_4)^2 I_3 K_B' - P_4''(L_4)^2 I_3 K_B'' \quad (20a)$$

$$L_4 I_3 M_4^4 + 2(L_4 I_5 + L_5 I_4)M_5^4 + 0 = -P_4'(L_4)^2 I_5 K_A' - P_4''(L_4)^2 I_5 K_A'' - 0 \quad (21a)$$

Solve for M_4^4 and M_5^4

For Fig. 3f

$$0 + 2(C_L L_3 I_4 + L_4 I_3)M_5^4 + L_4 I_3 M_5^5 = -0 - 0 - 0 \quad (22a)$$

$$L_4 I_5 M_5^4 + 2(L_4 I_5 + L_5 I_4)M_5^5 + 0 = -0 - P_5'(L_5)^2 I_4 K_B' - P_5''(L_5)^2 I_4 K_B'' \quad (23a)$$

Solve for M_5^5

In Eqs. (14a) to (23a) $K_A' = [k' - (k')^3]$; $K_A'' = [k'' - (k'')^3]$ and $K_B' = [2k' - 3(k')^2 + (k')^3]$; $K_B'' = [2k'' - 3(k'')^2 + (k'')^3]$

Having obtained the PARTIAL MOMENTS we are now ready for the next step, namely, the determination of the moments at every support over which the beam is continuous in terms of the partial moments. This is done by multiplying the Partial Moments by certain coefficients of MULTIPLIERS being careful to give them the proper signs in accordance with Proposition I. These multipliers will be designated by letters of the alphabet as follows: "a", "b", "c" etc. when expressing moments at given supports in terms of the moment at a succeeding support and "k", "l", "m" etc. when expressing moments at given supports in terms of the moment at a preceding support. Thus referring to

Fig. 1b

$$M_5' = +k \times l \times m \times M_2' \quad (24)$$

$$M_4' = -k \times l \times M_2' \quad (25)$$

$$M_3' = +k \times M_2' \quad (26)$$

Fig. 1c

$$M_5^2 = - \; l \times m \times M_3^2 \quad (27)$$

$$M_4^2 = + \; l \times M_3^2 \quad (28)$$

Fig. 1d

$$M_5^3 = + \; m \times M_4^3 \quad (29)$$

$$M_2^3 = + \; a \times M_3^3 \quad (30)$$

Fig. 1e

$$M_2^4 = - \; a \times b \times M_4^4 \quad (31)$$

$$M_3^4 = + \; b \times M_4^4 \quad (32)$$

Fig. 1f

$$M_2^5 = + a \times b \times c \times M_5^5 \quad (33)$$

$$M_3^5 = - \; b \times c \times M_5^5 \quad (34)$$

$$M_4^5 = + \; c \times M_5^5 \quad (35)$$

Formulas for obtaining the values of the "multipliers" are given on Plate IV.

The final step is the SUMMATION of the Partial Moments to obtain the Total Moments in accordance with Proposition II. Applying Eqs. (24) to (35) to Eqs. (10) to (13) we can now write the equations for the moments at the supports 2, 3, 4 and 5 in Fig. 1a.

$$M_2 = - \; M_2' - M_2^2 + a \times M_3^3 - a \times b \times M_4^4 + a \times b \times c \times M_5^5 \quad (36)$$

$$M_3 = + \; k \times M_2' - M_3^2 - M_3^3 + b \times M_4^4 - b \times c \times M_5^5 \quad (37)$$

$$M_4 = - \; k \times l \times M_2' + l \times M_3^2 - M_4^3 - M_4^4 + c \times M_5^5 \quad (38)$$

$$M_5 = + k \times l \times m \times M_2' + l \times m \times M_3^2 + m \times M_4^3 - M_5^4 - M_5^5 \quad (39)$$

Eqs. (36) to (39) give values of the moments at the supports precisely the same as do Eqs. (1) to (4), but the amount of time and labor involved is greatly reduced. This is especially true if the moments are to be expressed in terms of the loadings on the spans.

Illustrative Problems

Analytical Solution

The solution of the illustrative problems will be given in full detail, listing every step from the method of obtaining the constants C_r and C_l, multipliers a, b, c etc. to the combination of these multipliers in the summation of the Partial Moments. In actual practice it will be found that a number of short cuts can be used in obtaining and applying these factors without affecting the accuracy of the results too materially.

A list of short cuts is given at the end of the illustrative problems.

Illustrative Problem # 1 (Uniform Loading) solved in terms of "W".

Referring to Fig. 2a, Plate I.

Let spans $L_1 = L_5 = 18'$; $L_2 = L_4 = 24'$; $L_3 = 30'$

Let $I_1 = I_5 = 1000''^4$; $I_2 = I_4 = 2000''^4$; $I_3 = 3000''^4$

Let loading on $L_1 = W_1$; $L_2 = W_2$; $L_3 = W_3$; $L_4 = W_4$; $L_5 = W_5$

Step One—Subdivision of Beams

From Plate I,

$$C_L L_2 = 21.60'; \quad C_L L_3 = 26.55'; \quad C_R L_3 = 26.55'; \quad C_R L_4 = 21.60'$$

Step Two—Partial Moments

Applying Eqs. (14) to (23) we obtain

Fig. 2b $M_2^1 = 25.76\, W_1$

Fig. 2c $M_2^2 = 21.95\, W_2$; $M_3^2 = 35.70\, W_2$

Fig. 2d $M_3^3 = 43.65\, W_3$; $M_4^3 = 43.65\, W_3$

Fig. 2e $M_4^4 = 35.70\, W_4$; $M_5^4 = 21.95\, W_4$

Fig. 2f $M_5^5 = 25.76\, W_5$

Step Three—Summation of Partial Moments

From Plate IV get Multipliers

$$a = .20, b = .24, c = .29; k = .29, l = .24, m = .20$$

Follow the method of summation of Partial Moments given on Plate V and reduce to obtain Total Moments at the supports:

Total Moments in terms of "W".

$$M_2 = -25.76\, W_1 - 21.95\, W_2 + 8.73 W_3 - 1.72 W_4 + .36 W_5 \qquad (40)$$

$$M_3 = +7.50\, W_1 - 35.70\, W_2 - 43.65 W_3 + 8.60 W_4 - 1.80 W_5 \qquad (41)$$

$$M_4 = -1.80\, W_1 + 8.60\, W_2 - 43.65 W_3 - 35.70 W_4 + 7.50 W_5 \qquad (42)$$

$$M_5 = +.36\, W_1 - 1.72\, W_2 + 8.73 W_3 - 21.95 W_4 - 25.76 W_5 \qquad (43)$$

If the loading on any one of the five spans is changed it is only necessary to substitute the new value of "W" (W = 0 where loading is omitted) into Eqs. (40) to (43) to get the new moments. Therein lies the advantage of solving in terms of the loading "W".

Illustrative Problem # 2 (Concentrated Loading) Solved in terms of "P".

Referring to Fig. 3a, Plate II,

Let Span $L_1 = L_5 = 18.0'$; $L_2 = L_4 = 24.0'$; $L_3 = 30'$

Let $I_1 = I_5 = 1000''^4$; $I_2 = I_4 = 2000''^4$; $I_3 = 3000''^4$

Let loading on $L_1 = P_1' \,\&\, P_1''$; $L_2 = P_2' \,\&\, P_2''$; $L_3 = P_3' \,\&\, P_3''$; $L_4 = P_4' \,\&\, P_4''$; $L_5 = P_5' \,\&\, P_5''$

Step One—Subdivision of Beams

From Plate I,

$$C_L L_2 = 21.60', \; C_L L_3 = 26.55', \; C_R L_3 = 26.55', \; C_R L_4 = 21.60'$$

Step Two—Partial Moments

Applying Eqs. (14a) to (23a) we obtain

Fig. 3b $\quad M_2^1 = 1.48P_1' + 2.20P_1''$

Fig. 3c $\quad M_2^2 = 1.40P_2' + 1.32P_2''$; $\quad M_3^2 = 1.47P_2' + 2.26P_2''$

Fig. 3d $\quad M_3^3 = 2.07P_3' + 1.90P_3''$; $\quad M_4^3 = 1.35P_3' + 2.32P_3''$

Fig. 3e $\quad M_4^4 = 2.31P_4' + 1.81P_4''$; $\quad M_5^4 = 1.12P_4' + 1.49P_4''$

Fig. 3f $\quad M_5^5 = 2.20P_5' + 1.23P_5''$

Step Three—Summation of Partial Moments

From Plate IV get Multipliers

$$a = .20, b = .24, c = .29; k = .29, l = .24, m = .20$$

Follow the method of summation of Partial Moments given on Plate V and reduce to obtain Total Moments at the supports.

Total Moments in terms of "P":

$$M_2 = -1.48P_1' - 2.20P_1'' - 1.40P_2' - 1.32P_2'' + 0.41P_3' + 0.38P_3'' - 0.11P_4' - 0.09P_4'' + 0.03P_5' + 0.02P_5'' \quad (44)$$

$$M_3 = +0.43P_1' + 0.64P_1'' - 1.47P_2' - 2.26P_2'' - 2.07P_3' - 1.90P_3'' + 0.55P_4' + 0.43P_4'' - 0.15P_5' - 0.09P_5'' \quad (45)$$

$$M_4 = -0.10P_1' - 0.15P_1'' + 0.35P_2' + 0.54P_2'' - 1.35P_3' - 2.32P_3'' - 2.31P_4' - 1.81P_4'' + 0.64P_5' + 0.36P_5'' \quad (46)$$

$$M_5 = +0.02P_1' + 0.03P_1'' - 0.07P_2' - 0.10P_2'' + 0.27P_3' + 0.46P_3'' - 1.12P_4' - 1.49P_4'' - 2.20P_5' - 1.23P_5'' \quad (47)$$

If the loading on any one of the five spans is changed it is only necessary to substitute the new value of "P" (P = 0 where loading is omitted) into Eqs. (44) to (47) to get the new moments. Therein lies the advantage of solving in terms of the loading "P".

Illustrative Problems

Graphical Solution

In the analytical solution the Partial Moments (Step Two) were obtained by applying the Theorem of Three Moments to each subdivided beam having three spans, ends simply supported and loaded on one span only. In the Graphical solution the Partial Moments will be obtained from the graphs given on Charts VI to XVII.

In their construction the graphs are similar to, but are not identical with, those given in J. A. L. Waddell's[3] Bridge Engineering. Furthermore, the graphs developed by the writer are applicable to both concentrated and uniform loading, whereas Waddell's graphs are applicable to uniform loading only.

The solution of the illustrative problems given below will describe the method in detail. As explained in the case of the analytical solution, a number of short-cuts are available to the engineer using this method in actual practice. A number of such short-cuts are given at the end of the illustrative problems.

Illustrative Problem # 1 (uniform loading) solved in terms of "W"

Referring to Fig. 2a, Plate I

Let spans $L_1 = L_5 = 18'$, $L_2 = L_4 = 24'$ and $L_3 = 30'$

Let $I_1 = I_5 = 1000''^4$, $I_2 = I_4 = 2000''^4$ and $I_3 = 3000''^4$

Let the loading on $L_1 = W_1$, $L_2 = W_2$, $L_3 = W_3$, $L_4 = W_4$ and $L_5 = W_5$

Step One—Subdivision of Beam

From Plate I,

$$C_L L_2 = 21.60', \; C_L L_3 = 26.55, \; C_R L_3 = 26.55', \; C_R L_4 = 21.60'$$

Step Two—Partial Moments

(a) Obtain f_1 and f_2

Fig. 2b	$f_1 = .67$,	$f_2 = 1.36$	(Plate VI)
Fig. 2c	$f_1 = .67$,	$f_2 = 1.36$	(Plate IX)
Fig. 2d	$f_1 = .93$,	$f_2 = .93$	(Plate IX)
Fig. 2e	$f_1 = 1.36$,	$f_2 = .67$	(Plate X)
Fig. 2f	$f_1 = 1.36$,	$f_2 = .67$	(Plate VII)

(b) Read Curves "A" and "B".

These curves give the coefficients "A" and "B" which enter the expression for moments $AW_n(L_n)^2$ and $BW_n(L_n)^2$. In the END spans coefficient "A" applies to the moment at the RIGHT support and coefficient "B" to the moment at the LEFT support. In the INTERMEDIATE spans coefficient "A" applies to the moment at the LEFT support and coefficient "B" to the moment at the RIGHT support (See Plates VI to XI incl.)

To read the curves we enter the abscissas with the value of f_1, up vertically to curves "A" or "B" to an intersection with the corresponding value of f_2, then to the left or right where we read the coefficients "A" or "B".

Thus we read

Fig. 2b	A = .0795		(Plate VI)
Fig. 2c	A = .0380	B = .0620	(Plate IX)
Fig. 2d	A = .0485	B = .0485	(Plate IX)

Fig. 2e $A = .0620$ $B = .0380$ (Plate X)

Fig. 2f $B = .0795$ (Plate VII)

(c) Write Partial Moments

Fig. 2b $M_2^1 = .0795 W_1 (L_1)^2 = 25.76 W_1$

Fig. 2c $M_2^2 = .0380 W_2 (L_2)^2 = 21.95 W_2$; $M_3^2 = .0620 W_2 (L_2)^2 = 35.70 W_2$

Fig. 2d $M_3^3 = .0485 W_3 (L_3)^2 = 43.65 W_3$; $M_4^3 = .0485 W_3 (L_3)^2 = 43.65 W_3$

Fig. 2e $M_4^4 = .0620 W_4 (L_4)^2 = 35.70 W_4$; $M_5^4 = .0380 W_4 (L_4)^2 = 21.95 W_4$

Fig. 2f $M_5^5 = .0795 W_5 (L_5)^2 = 25.76 W_5$

Step Three—Summation of Partial Moments

(a) Get Multipliers (Plate IV)

$a = .20$; $b = .24$; $c = .29$

$k = .29$; $l = .24$; $m = .20$

(b) Follow the method of summation of Partial Moments given on Plate V and reduce: to obtain Total Moments at the supports.

(c) Total Moments in terms of "W"

$$M_2 = -25.76 W_1 - 21.95 W_2 + 8.73 W_3 - 1.72 W_4 + 0.36 W_5 \quad (40)$$

$$M_3 = +7.50 W_1 - 35.70 W_2 - 43.65 W_3 + 8.60 W_4 - 1.80 W_5 \quad (41)$$

$$M_4 = -1.80 W_1 + 8.60 W_2 - 43.65 W_3 - 35.70 W_4 + 7.50 W_5 \quad (42)$$

$$M_5 = +0.36 W_1 - 1.72 W_2 + 8.73 W_3 - 21.95 W_4 - 25.76 W_5 \quad (43)$$

If the loading on any one span is changed or omitted it is only necessary to substitute the new values for "W" (W = 0, where the load is omitted) into Eqs. (40) to (43) to get the required moments. Therein lies the advantage of solving in terms of the loading "W".

Illustrative Problem # 2, Concentrated Loading (Solving in terms of "P")

Referring to Fig. 3a, Plate II

Let Spans $L_1 = L_5 = 18'$, $L_2 = L_4 = 24'$ and $L_3 = 30'$

Let $I_1 = I_5 = 1000''^4$, $I_2 = I_4 = 2000''^4$ and $I_3 = 3000''^4$

Let Loading on $L_1 = P_1' \& P_1''$, $L_2 = P_2' \& P_2''$, $L_3 = P_3' \& P_3''$, $L_4 = P_4' \& P_4''$ and $L_5 = P_5' \& P_5''$

Step One—Subdivision of Beam

From Plate I

$C_L L_2 = 21.60'$, $C_L L_3 = 26.55$, $C_R L_3 = 26.55'$, $C_R L_4 = 21.60'$

Step Two—Partial Moments

In the ANALYTICAL method the Partial Moments are found by applying Eqs. (14a) to (23a), incl. In the GRAPHICAL method proceed as follows:

(a) Obtain f_1 and f_2

Fig. 3b	$f_1 = .67$, $f_2 = 1.36$	(Plate XII)
Fig. 3c	$f_1 = .67$, $f_2 = 1.36$	(Plate XV)
Fig. 3d	$f_1 = .93$, $f_2 = .93$	(Plate XV)
Fig. 3e	$f_1 = 1.36$, $f_2 = .67$	(Plate XVI)
Fig. 3f	$f_1 = 1.36$, $f_2 = .67$	(Plate XIII)

(b) Read curves "A", "B", "K_a" and "K_b".

These curves give the coefficients "A", "B", "K_a" and "K_b" which enter the expressions for moments as follows: In the END spans $-A L_n \leq (K_A P_n)$ for the RIGHT support and $B L_n \leq (K_B P_n)$ for the LEFT support. (See Plates XII to XIV). In the INTERMEDIATE spans $A L_n \leq (K_B P_n)$ for the LEFT support and $B L_n \leq (K_A P_n)$ for the RIGHT support. (See Plates XV to XVII).

To read the curves "A" and "B" we enter the abscissas with the value of f_1, up vertically to curve "A" or "B" to an intersection with the corresponding value of f_2, then left or right where we read the coefficients "A" or "B".

To read the curves K_a and K_b we enter the abscissas with the value of k for each concentrated load, down vertically to an intersection with curves K_a or K_b, then to the left or right where we read the coefficients K_a or K_b.

Thus we read:

Fig. 3b	$A = 0.318$			(Plate XII)
Fig. 3c	$A = 0.155$		$B = 0.245$	(Plate XV)
Fig. 3d	$A = 0.195$		$B = 0.195$	(Plate XV)
Fig. 3e	$A = 0.245$		$B = 0.155$	(Plate XVI)
Fig. 3f			$B = 0.318$	(Plate XIII)
And	$k' = 0.278$		$K'_A = 0.260$	
Fig. 3b	$k'' = 0.611$		$K''_A = 0.385$	(Plate XII)
	$k' = 0.291$	$K'_B = 0.350$	$K'_A = 0.265$	(Plate XII)
Fig. 3c	$k'' = 0.541$	$K''_B = 0.360$	$K''_A = 0.380$	(Plate XV)
	$k' = 0.267$	$K'_B = 0.335$	$K'_A = 0.245$	(Plate XV)
Fig. 3d	$k'' = 0.600$	$K''_B = 0.335$	$K''_A = 0.385$	(Plate XV)
	$k' = 0.375$	$K'_B = 0.380$	$K'_A = 0.320$	(Plate XVI)
Fig. 3e	$k'' = 0.625$	$K''_B = 0.320$	$K''_A = 0.380$	(Plate XVI)
Fig. 3f	$k' = 0.444$	$K'_B = 0.385$		(Plate XIII)
	$k'' = 0.777$	$K''_B = 0.215$		(Plate XIII)

(c) Write Partial Moments:

Fig. 3b $M_2^1 = .318 L_1 (.260 P_1' + .385 P_1'')$

Fig. 3c $M_2^2 = .155 L_2 (.350 P_2' + .360 P_2'')$; $M_3^2 = .245 L_2 (.265 P_2' + .380 P_2'')$

Fig. 3d $M_3^3 = .195 L_3 (.335 P_3' + .335 P_3'')$; $M_4^3 = .195 L_3 (.245 P_3' + .385 P_3'')$

Fig. 3e $M_4^4 = .245 L_4 (.380 P_4' + .320 P_4'')$; $M_5^4 = .155 L_4 (.320 P_4' + .380 P_4'')$

Fig. 3f $M_5^5 = .318 L_5 (.385 P_5' + .215 P_5'')$;

(d) Reduce:

Fig. 3b $M_2' = 1.48 P_1' + 2.20 P_1''$

Fig. 3c $M_2^2 = 1.30 P_2' + 1.34 P_2''$; $M_3^2 = 1.57 P_2' + 2.24 P_2''$

Fig. 3d $M_3^3 = 1.96 P_3' + 1.96 P_3''$; $M_4^3 = 1.46 P_3' + 2.26 P_3''$

Fig. 3e $M_4^4 = 2.24 P_4' + 1.88 P_4''$; $M_5^4 = 1.19 P_4' + 1.42 P_4''$

Fig. 3f $M_5^5 = 2.20 P_5' + 1.23 P_5''$

Step Three—Summation of Partial Moments:

(a) Get Multipliers (Plate IV)

$$a = .20 \quad , \quad b = .24 \quad , \quad c = .29$$
$$k = .29 \quad , \quad l = .24 \quad , \quad m = .20$$

(b) Follow the method of summation of Partial Moments given on Plate V and reduce to obtain total moments at the supports.

(c) Total Moments in terms of "P"

$$M_2 = -1.48 P_1' - 2.20 P_1'' - 1.30 P_2' - 1.34 P_2'' + 0.39 P_3' + 0.39 P_3'' - 0.11 P_4' - 0.09 P_4'' + 0.03 P_5' + 0.02 P_5'' \quad (48)$$

$$M_3 = +0.43 P_1' + 0.64 P_1'' - 1.57 P_2' - 2.24 P_2'' - 1.96 P_3' - 1.96 P_3'' + 0.54 P_4' + 0.45 P_4'' - 0.15 P_5' - 0.09 P_5'' \quad (49)$$

$$M_4 = -0.10 P_1' - 0.15 P_1'' + 0.38 P_2' + 0.54 P_2'' - 1.46 P_3' - 2.26 P_3'' - 2.24 P_4' - 1.88 P_4'' + 0.64 P_5' + 0.36 P_5'' \quad (50)$$

$$M_5 = +0.02 P_1' + 0.03 P_1'' - 0.08 P_2' - 0.11 P_2'' + 0.29 P_3' + 0.45 P_3'' - 1.19 P_4' - 1.42 P_4'' - 2.20 P_5' - 1.23 P_5'' \quad (51)$$

If the loading on any one of the five spans is changed it is only necessary to substitute the new value of "P" (P = 0 where the loading is omitted) into Eqs. (48) to (51) to get the new moments. Therein lies the advantage of solving in terms of "P".

Short-Cuts

Attention is directed to the following short-cuts which may be used if a lesser degree of accuracy in the results obtained is deemed justified by the engineer.

Constants C_r and C_l

An examination of the Eqs. (Plate I)

$$C_R = 1 - \frac{L_n I_{n+1}}{4(L_n I_{n+1} + L_{n+1} I_n)} \quad \text{and} \quad C_L = 1 - \frac{L_n I_{n-1}}{4(L_{n-1} I_n + L_n I_{n-1})}$$

indicates that if the lengths of all the spans are equal and the moments of inertia of all the beams are equal, then $C_r = C_l = 0.875$ (7/8). Again, it will be noted from the illustrative problems that for spans of unequal length and for beams having different moments of inertia these constants vary from 0.885 to 0.900. It is, therefore, apparent that a value of 0.875(7/8) for both C_r and C_l may be used without a material loss of accuracy in the results. A word of caution must be added here, however, to the effect that in extreme cases of variation in span lengths and moments of inertia the value of these constants may vary from 0.75 to 1.00.

Multipliers a, b, c, and k. l, m etc.

An examination of the Eqs. (Plate IV)

$$a, b, c \text{ etc.} = \frac{L_n I_{n-1}}{2(L_{n-1} I_n + L_n I_{n-1})} \quad \text{and} \quad k, l, m \text{ etc.} = \frac{L_n I_{n+1}}{2(L_n I_{n+1} + L_{n+1} I_n)}$$

indicates that if the spans are equal in length and the moments of inertia of the beam for all the spans are also equal, then each of the multipliers will be equal to 0.25 (1/4). On the other hand, the illustrative problems show that when the span lengths and moments of inertia are not equal the values of the multipliers vary from 0.20 to 0.29. The error resulting from neglecting this variation in the values of the multipliers is appreciable as may be observed in the following tabulation:

Using the value of 0.25(1/4) for all the multipliers in our problems

$a = \frac{1}{4}$, $a \times b = \frac{1}{16}$, $a \times b \times c = \frac{1}{64}$ (Similar for k, l, m etc.)

Using the correct values of the multipliers,

$a = \frac{1}{5}$, $a \times b = \frac{1}{21}$, $a \times b \times c = \frac{1}{72}$ (Similar for k, l, m etc.)

The engineer's judgment must guide him in the choice of this short-cut.

Summation of Partial Moments

If more than 5 spans are involved time and labor may be saved without any material sacrifice of accuracy if we neglect the terms involving more than 3 multipliers. Thus, referring to Plate V, in the 6-span beam, we may disregard the terms

$$. a \times b \times c \times d \; M_6^6 \; (\text{For } M_2) \text{ and } k \times l \times m \times n \; M_2' \; (\text{For } M_6)$$

and in the 7-span beam we may disregard the terms

$$a \times b \times c \times d \; M_6^6 \; (\text{For } M_2);$$

$$. a \times b \times c \times d \times e \; M_7^7 (\text{For } M_2); \; b \times c \times d \times e \; M_7^7 (\text{For } M_3); \; k \times l \times m \times n \; M_2' \; (\text{For } M_6);$$

$$. k \times l \times m \times n \times o \; M_2' \; (\text{For } M_7) \text{ and } l \times m \times n \times o \; M_3^2 \; (\text{For } M_7$$

Correction (Optional) for the Graphical Solution

It will be noted that the Partial Moments obtained by the graphical method do not quite agree with those obtained by the analytical method. However, the resultant error in the Total Moments due to this discrepancy is very small and may be safely neglected. As a matter of record, however, the nature of and the method of correcting this error will now be explained.

Due to the mechanics of construction of the curves "A" and "B" for concentrated loading (Plates VI to XVII) a small correction must be applied to the Partial Moments in the INTERMEDIATE spans (Figs. 3c to 3e). The correction $(K_b - K_a)\Delta K$ is to be applied to moments at the LEFT support and $(K_a - K_b)\Delta K$ is to be applied to moments at the RIGHT support.

To read the curves ΔK we enter the abscissas with the value f_1, up vertically to an intersection with the corresponding value of f_2, then to the right where we read the coefficients ΔK. Thus,

For loads P'

		Left Support $(K'_B - K'_A)\Delta k$	Right Support $(K'_A - K'_B)\Delta k$	
Fig. 3c	$f_1 = 0.67, f_2 = 1.36$	$+0.085 \times 1.2$	-0.085×1.2	(Plate XV)
Fig. 3d	$f_1 = 0.93, f_2 = 0.93$	$+0.090 \times 1.2$	-0.090×1.2	(Plate XV)
Fig. 3e	$f = 1.36, f_2 = 0.67$	$+0.060 \times 1.2$	-0.060×1.2	(Plate XVI)

For loads P"

		$(K''_B - K''_A)\Delta k$	$(K''_A - K''_B)\Delta k$	
Fig. 3c	$f_1 = 0.67, f_2 = 1.36$	-0.020×1.2	$+0.020 \times 1.2$	(Plate XV)
Fig. 3d	$f_1 = 0.93, f_2 = 0.93$	-0.050×1.2	$+0.050 \times 1.2$	(Plate XV)
Fig. 3e	$f_1 = 1.36, f_2 = 0.67$	-0.060×1.2	$+0.060 \times 1.2$	(Plate XVI)

Apply Corrections

Fig. 3b (No corrections.) $M'_2 = 1.48P'_1 + 2.20P''_1$

Fig. 3c $M^2_2 = (1.30 + .10)P'_2 + (1.34 - .02)P''_2$; $M^2_3 = (1.57 - .10)P'_2 + (2.24 + .02)P''_2$

Fig. 3d $M^3_3 = (1.96 + .11)P'_3 + (1.96 - .06)P''_3$; $M^3_4 = (1.46 - .11)P'_3 + (2.26 + .06)P''_3$

Fig. 3e $M^4_4 = (2.24 + .07)P'_4 + (1.88 - .07)P''_4$; $M^5_5 = (1.19 - .07)P'_4 + (1.42 + .07)P''_4$

Fig. 3f $M^5_5 = 2.20P'_5 + 1.23P''_5$ (No corrections)

Reduce

Fig. 3b $M'_2 = 1.48P'_1 + 2.20P''_1$

Fig. 3c $M^2_2 = 1.40P'_2 + 1.32P''_2$; $M^2_3 = 1.47P'_2 + 2.26P''_2$

Fig. 3d $M^3_3 = 2.07P'_3 + 1.90P''_3$; $M^3_4 = 1.35P'_3 + 2.32P''_3$

Fig. 3e $M^4_4 = 2.31P'_4 + 1.81P''_4$; $M^4_5 = 1.12P'_4 + 1.49P''_4$

Fig. 3f $M^5_5 = 2.20P'_5 + 1.23P''_5$

Step Three

Proceed as outlined in the preceding solution.

Special Cases

The method of Partial Moments is applicable to many special cases. Some of these are given on Plate III. The following cases are listed.

One Span, One End Fixed

Add one imaginary span $L_0 = 0$ at the fixed end and another at the free end $L_2 = \alpha$(infinity), then proceed as for a 3-span beam.

One Span, Both Ends Fixed

Add imaginary spans $L_0 = 0$ at one end and $L_2 = 0$ at the other, then proceed as for a 3-span beam.

Two Spans, Ends Free

Add one imaginary span $L_3 = \alpha$, then proceed as for a 3-span beam.

Two Spans, One End Fixed

Add one imaginary span $L_3 = 0$, then proceed as for a 3-span beam.

Two Spans, Both Ends Fixed

Add imaginary spans $L_0 = 0$ at one end and $L_3 = 0$ at the other, then proceed as for a 4-span beam.

"N" Spans, One End Fixed

Add one imaginary span $L_{n+1} = 0$ at the fixed end, then proceed as in the normal case.

"N" Spans, Both Ends Fixed

Add imaginary spans L_0 and $L_{n+1} = 0$ at each end, then proceed as in the normal case.

"N" Spans, with Cantilevers at One or Both Ends

Proceed as in the normal case. Use the cantilever moments as Partial Moments $M_1{}^c$ and $M_{n+1}{}^c$ in writing the summation of Partial Moments.

Moments and Shears

Moments

In the foregoing matter we have developed the method of obtaining the moments at the supports of a continuous beam. Once these moments have been found, the moments at any point in the beam and the shears and reactions at the supports can be readily arrived at.

For any given span AB in which fixing moments M_a and M_b are found to exist at points A and B respectively, the moments at any point distant "x" from B will be

$$M = M_1 + M_b + (M_a - M_b)\frac{x}{L} \qquad (1)$$

where M_1 is the bending moment for a freely supported beam of the same span at a distance "x" from B and M_a and M_b are the moments at the supports A and B respectively.

Shears

The shear at either end of any span, as determined by the law of the lever, must be increased or decreased by an amount found by dividing the difference of the moments at the extremities of any span by the span itself.

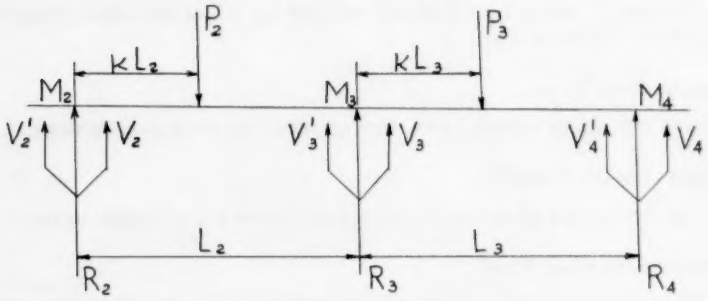


FIG. 1

Referring to Fig. 1, for the shears "V" there may be written:

$$V_2 = P_2\left(\frac{L_2 - kL_2}{L_2}\right) - \frac{M_2 - M_3}{L_2} \qquad (2)$$

$$V'_3 = P_2\left(\frac{kL_2}{L_2}\right) + \frac{M_2 - M_3}{L_2} \qquad (3)$$

$$V_3 = P_3\left(\frac{L_3 - kL_3}{L_3}\right) - \frac{M_3 - M_4}{L_3} \qquad (4)$$

$$V'_4 = P_3\left(\frac{kL_3}{L_3}\right) + \frac{M_3 - M_4}{L_3} \qquad (5)$$

It is important to note the sign before the fractions

$$\frac{M_2 - M_3}{L_2} \quad \text{and} \quad \frac{M_3 - M_4}{L_3}$$

In Eqs. (2) and (4) the signs are negative, while in Eqs. (3) and (5) they are positive. The reason for this is to be found in the fact that, while in Fig. 1 the moments M_2 and M_3 have opposite signs to those caused by the loadings P_2 and P_3, in Eqs. (1) the three component moments are given the same sign.

Another method for obtaining the shears "V_2" etc., may be resorted to by taking moments about either end of the spans L_2 and L_3 in Fig. 1. Thus, taking moments about the right end of L_2,

$$V_2 \times L_2 - \Sigma P_2 \left(\frac{L_2 - kL_2}{L_2}\right) + M_2 = M_3 \quad (6)$$

Taking moments about the left end of L_2,

$$V_3' \times L_2 - \Sigma P_2 \left(\frac{kL_2}{L_2}\right) + M_3 = M_2 \quad (7)$$

Again, taking moments about the right end of L_3,

$$V_3 \times L_3 - \Sigma P_3 \left(\frac{L_3 - kL_3}{L_3}\right) + M_3 = M_4 \quad (8)$$

And taking moments about the left end of L_3,

$$V_4' \times L_3 - \Sigma P_3 \left(\frac{kL_3}{L_3}\right) + M_4 = M_3 \quad (9)$$

From Eqs. (6), (7), (8) and (9) the Eqs. (2), (3), (4) and (5) at once result.

Thus it will be seen that by the aid of Eqs. (1) the moments at any point of a continuous beam may be determined. Similarly, by the aid of Eqs. (2), (3), (4) and (5) the shears at any support may be determined. The reaction at any support is equal to the sum of the shears at that point.

In Fig. 1,

$$R_2 = V_2' + V_2 \; ; \; R_3 = V_3' + V_3 \text{ and } V_4' + V_4$$

Stiff Frames

The method of Partial Moments may be applied to Stiff Frames under the following conditions:

1. In the case of L-shaped frames, for any type of loading.
2. In the case of one-story, one span frames and of one-story, multi span frames, the loading must be placed symmetrically about a vertical center line and the frame itself must be symmetrical about the same center line.

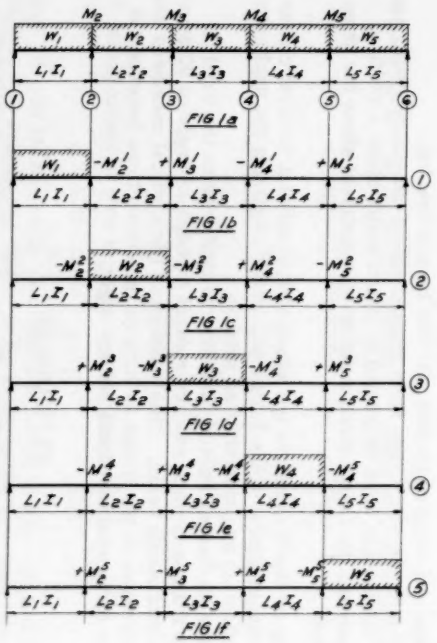
If the loading or the frames are not symmetrical about a vertical center line, a lateral displacement of the frame will occur requiring the application of the slope-deflection or other similar method.

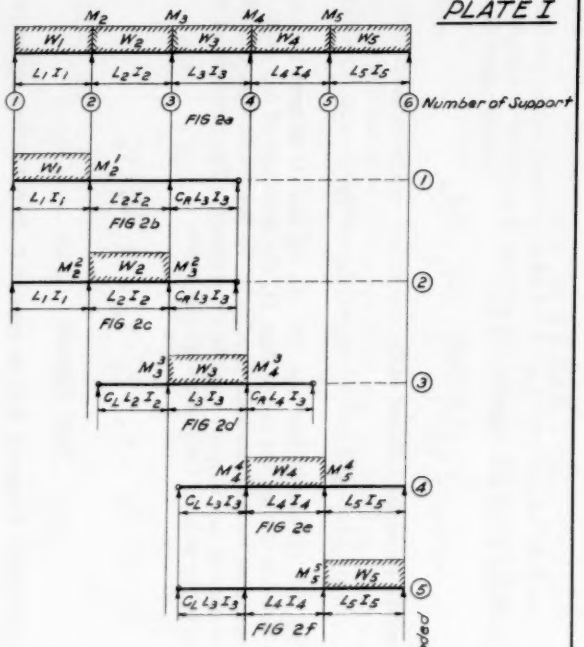
In applying the Method of Partial Moments to stiff frames the vertical members of the frames at each end are rotated through an angle of 90 degrees, thus forming a continuous beam. This rotation of the vertical members does not affect the stresses in the frame as long as the connections between the vertical and the horizontal members remain rigid.

REFERENCES

1. Mishkian, L. N. and Steinman, D. B., The Method of Conjugate Points, Proceedings ASCE, October, 1925.
2. Cross, Hardy, The Method of Distributing Fixed-End Moments, Proceedings ASCE, May, 1930.
3. Waddell, J. A. L., Bridge Engineering, John Wiley & Sons Inc., 1916, Volume I.

PLATE I





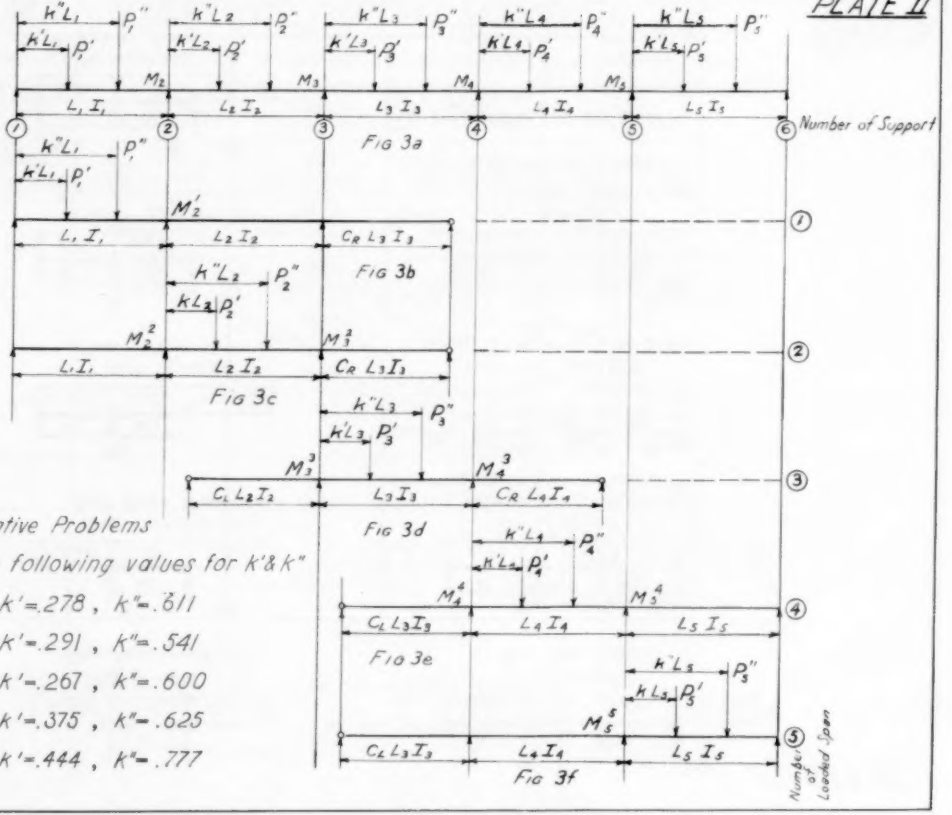
VALUES OF C_R and C_L

$$C_R = 1 - \frac{L_n I_{n+1}}{4(L_n I_{n+1} + L_{n+1} I_n)}$$

$$C_L = 1 - \frac{L_n I_{n-1}}{4(L_{n-1} I_n + L_n I_{n-1})}$$

Where "n" refers to the Number of the shortened Span($L_2, L_3, L_4 \ldots$)
" "I" = Moment of Inertia of Beam

PLATE II



NOTE:

For Illustrative Problems assume the following values for k' & k"

Fig. 3b – k' = .278 , k" = .611

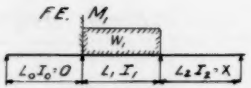
Fig. 3c – k' = .291 , k" = .541

Fig. 3d – k' = .267 , k" = .600

Fig. 3e – k' = .375 , k" = .625

Fig. 3f – k' = .444 , k" = .777

PLATE III

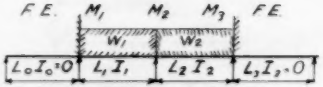


ONE SPAN
One End Fixed

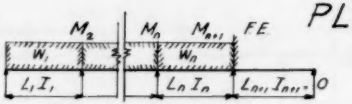
ONE SPAN
Both Ends Fixed

TWO SPANS
Ends Free

TWO SPANS
One End Fixed

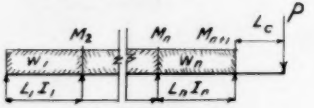


TWO SPANS
Both Ends Fixed



"n" SPANS
One End Fixed

"n" SPANS
Both Ends Fixed



"n" SPANS
Cantilever One End
$M_{n+1} = P \times L_c$

"n" SPANS
Cantilever Both Ends
M_1 and $M_{n+1} = P \times L_c$

F.E. designate Fixed End
∞ = Infinity

SPECIAL CASES

MULTIPLIERS FOR PARTIAL MOMENTS

PLATE IV

3-SPANS	4-SPANS	5-SPANS	6-SPANS	7-SPANS
$a = \frac{L_2 I_1}{2(L_1 I_2 + L_2 I_1)}$	$a = \frac{L_2 I_1}{2(L_1 I_2 + L_2 I_1)}$	$a = \frac{L_2 I_1}{2(L_1 I_2 + L_2 I_1)}$	$a = \frac{L_2 I_1}{2(L_1 I_2 + L_2 I_1)}$	$a = \frac{L_2 I_1}{2(L_1 I_2 + L_2 I_1)}$
$k = \frac{L_2 I_3}{2(L_2 I_3 + L_3 I_2)}$	$b = \frac{L_3 I_2}{2(C_L L_2 I_3 + L_3 I_2)}$	$b = \frac{L_3 I_2}{2(C_L L_2 I_3 + L_3 I_2)}$	$b = \frac{L_3 I_2}{2(C_L L_2 I_3 + L_3 I_2)}$	$b = \frac{L_3 I_2}{2(C_L L_2 I_3 + L_3 I_2)}$
	$k = \frac{L_2 I_3}{2(L_2 I_3 + C_R L_3 I_2)}$	$c = \frac{L_4 I_3}{2(C_L L_3 I_4 + L_4 I_3)}$	$c = \frac{L_4 I_3}{2(C_L L_3 I_4 + L_4 I_3)}$	$c = \frac{L_4 I_3}{2(C_L L_3 I_4 + L_4 I_3)}$
	$l = \frac{L_3 I_4}{2(L_3 I_4 + L_4 I_3}$	$k = \frac{L_2 I_3}{2(L_2 I_3 + C_R L_3 I_2)}$	$d = \frac{L_5 I_4}{2(C_L L_4 I_5 + L_5 I_4}$	$d = \frac{L_5 I_4}{2(C_L L_4 I_5 + L_5 I_4)}$
		$l = \frac{L_3 I_4}{2(L_3 I_4 + C_R L_4 I_3)}$	$k = \frac{L_2 I_3}{2(L_2 I_3 + C_R L_3 I_2)}$	$e = \frac{L_6 I_5}{2(C_L L_5 I_6 + L_6 I_5)}$
		$m = \frac{L_4 I_5}{2(L_4 I_5 + L_5 I_4)}$	$l = \frac{L_3 I_4}{2(L_3 I_4 + C_R L_4 I_3)}$	$k = \frac{L_2 I_3}{2(L_2 I_3 + C_R L_3 I_2)}$
			$m = \frac{L_4 I_5}{2(L_4 I_5 + C_R L_5 I_4)}$	$l = \frac{L_3 I_4}{2(L_3 I_4 + C_R L_4 I_3)}$
			$n = \frac{L_5 I_6}{2(L_5 I_6 + L_6 I_5)}$	$m = \frac{L_4 I_5}{2(L_4 I_5 + C_R L_5 I_4)}$
				$n = \frac{L_5 I_6}{2(L_5 I_6 + C_R L_6 I_5)}$
				$o = \frac{L_6 I_7}{2(L_6 I_7 + L_7 I_6}$

Values of Multipliers

$$a, b, c, d, e = \frac{L_n I_{n-1}}{2(L_{n-1} I_n + L_n I_{n-1})} \quad ; \quad k, l, m, n, o = \frac{L_n I_{n+1}}{2(L_n I_{n+1} + L_{n+1} I_n)}$$

where "n" refers to the number of the shortened Span
" "I" " " " Moment of Inertia of Beam

SUMMATION OF PARTIAL MOMENTS

PLATE V

$$M_2 = -M_2^1 - M_2^2 + \alpha_1 M_3^2$$
$$M_3 = +K_1 M_2^1 - M_3^2 - M_3^3$$

3-SPANS

$$M_2 = - M_2^1 - M_2^2 + \alpha_1 M_3^2 - \alpha_1 b_1 M_4^3$$
$$M_3 = + K_1 M_2^1 - M_3^2 - M_3^3 + b_1 M_4^3$$
$$M_4 = - K_1 \lambda_1 M_2^1 + \lambda_1 M_3^2 - M_4^3 - M_4^4$$

4-SPANS

$$M_2 = - M_2^1 - M_2^2 + \alpha_1 M_3^2 - \alpha_1 b_1 M_4^3 + \alpha_1 b_1 c_1 M_5^4$$
$$M_3 = + K_1 M_2^1 - M_3^2 - M_3^3 + b_1 M_4^3 - b_1 c_1 M_5^4$$
$$M_4 = - K_1 \lambda_1 M_2^1 + \lambda_1 M_3^2 - M_4^3 - M_4^4 + c_1 M_5^4$$
$$M_5 = + K_1 \lambda_1 m_1 M_2^1 - \lambda_1 m_1 M_3^2 + m_1 M_4^3 - M_5^4$$

5-SPANS

$$M_2 = - M_2^1 - M_2^2 + \alpha_1 M_3^2 - \alpha_1 b_1 M_4^3 + \alpha_1 b_1 c_1 M_5^4 - \alpha_1 b_1 c_1 d_1 M_6^5$$
$$M_3 = + K_1 M_2^1 - M_3^2 - M_3^3 + b_1 M_4^3 - b_1 c_1 M_5^4 + b_1 c_1 d_1 M_6^5$$
$$M_4 = - K_1 \lambda_1 M_2^1 + \lambda_1 M_3^2 - M_4^3 - M_4^4 + c_1 M_5^4 - c_1 d_1 M_6^5$$
$$M_5 = + K_1 \lambda_1 m_1 M_2^1 - \lambda_1 m_1 M_3^2 + m_1 M_4^3 - m_1 M_5^4 - M_5^4 + d_1 M_6^5$$
$$M_6 = - K_1 \lambda_1 m_1 n_1 M_2^1 + \lambda_1 m_1 n_1 M_3^2 - m_1 n_1 M_4^3 + n_1 M_5^4 - M_6^4 - M_6^5$$

6-SPANS

$$M_2 = - M_2^1 - M_2^2 + \alpha_1 M_3^2 - \alpha_1 b_1 M_4^3 + \alpha_1 b_1 c_1 M_5^4 - \alpha_1 b_1 c_1 d_1 M_6^5 + \alpha_1 b_1 c_1 d_1 e_1 M_7^6$$
$$M_3 = + K_1 M_2^1 - M_3^2 - M_3^3 + b_1 M_4^3 - b_1 c_1 M_5^4 + b_1 c_1 d_1 M_6^5 - b_1 c_1 d_1 e_1 M_7^6$$
$$M_4 = - K_1 \lambda_1 M_2^1 + \lambda_1 M_3^2 - M_4^3 - M_4^4 + c_1 M_5^4 - c_1 d_1 M_6^5 + c_1 d_1 e_1 M_7^6$$
$$M_5 = + K_1 \lambda_1 m_1 M_2^1 - \lambda_1 m_1 M_3^2 + m_1 M_4^3 - M_5^4 - M_5^4 + d_1 M_6^5 - d_1 e_1 M_7^6$$
$$M_6 = - K_1 \lambda_1 m_1 n_1 M_2^1 + \lambda_1 m_1 n_1 M_3^2 - m_1 n_1 M_4^3 + n_1 M_5^4 - M_6^4 - M_6^5 + e_1 M_7^6$$
$$M_7 = + K_1 \lambda_1 m_1 n_1 o_1 M_2^1 - \lambda_1 m_1 n_1 o_1 M_3^2 + m_1 n_1 o_1 M_4^3 - n_1 o_1 M_5^4 + o_1 M_6^5 - M_7^6 - M_7^7$$

7-SPANS

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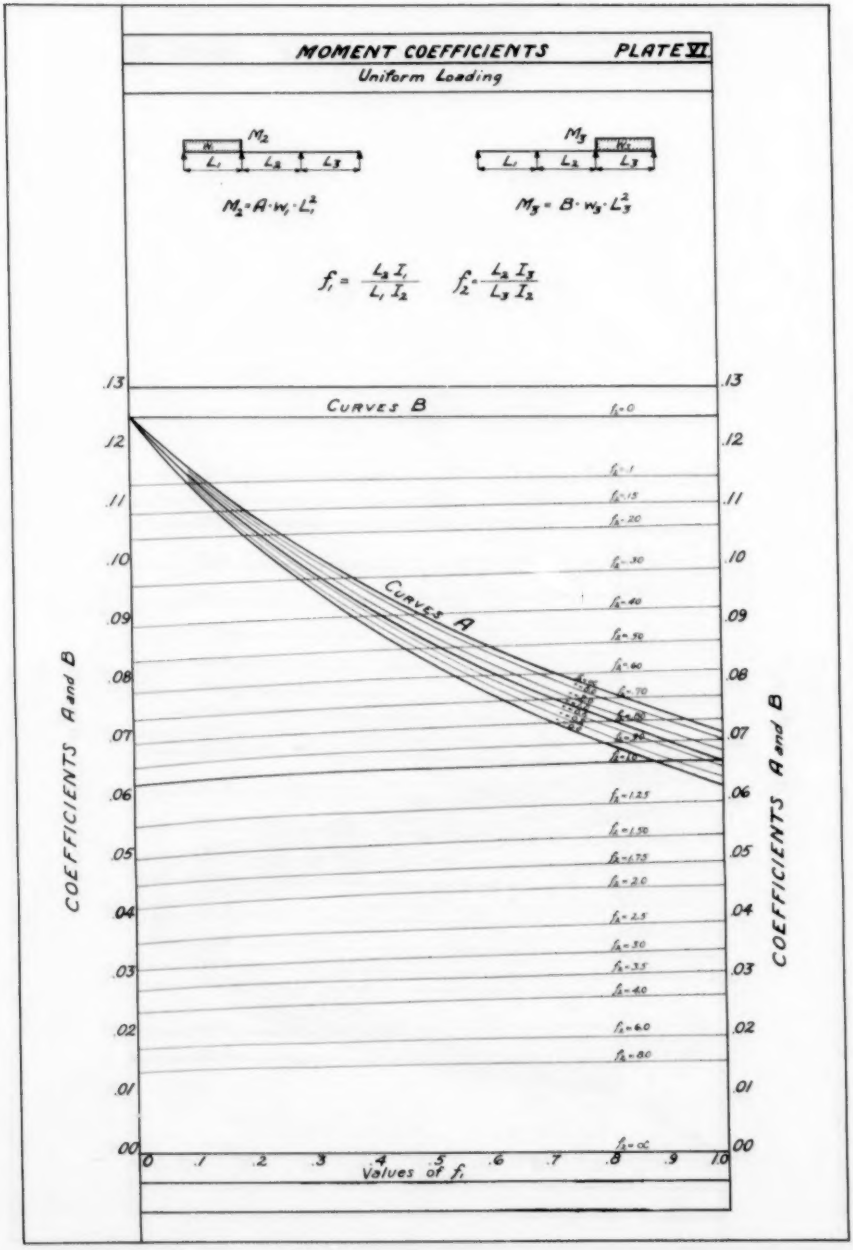
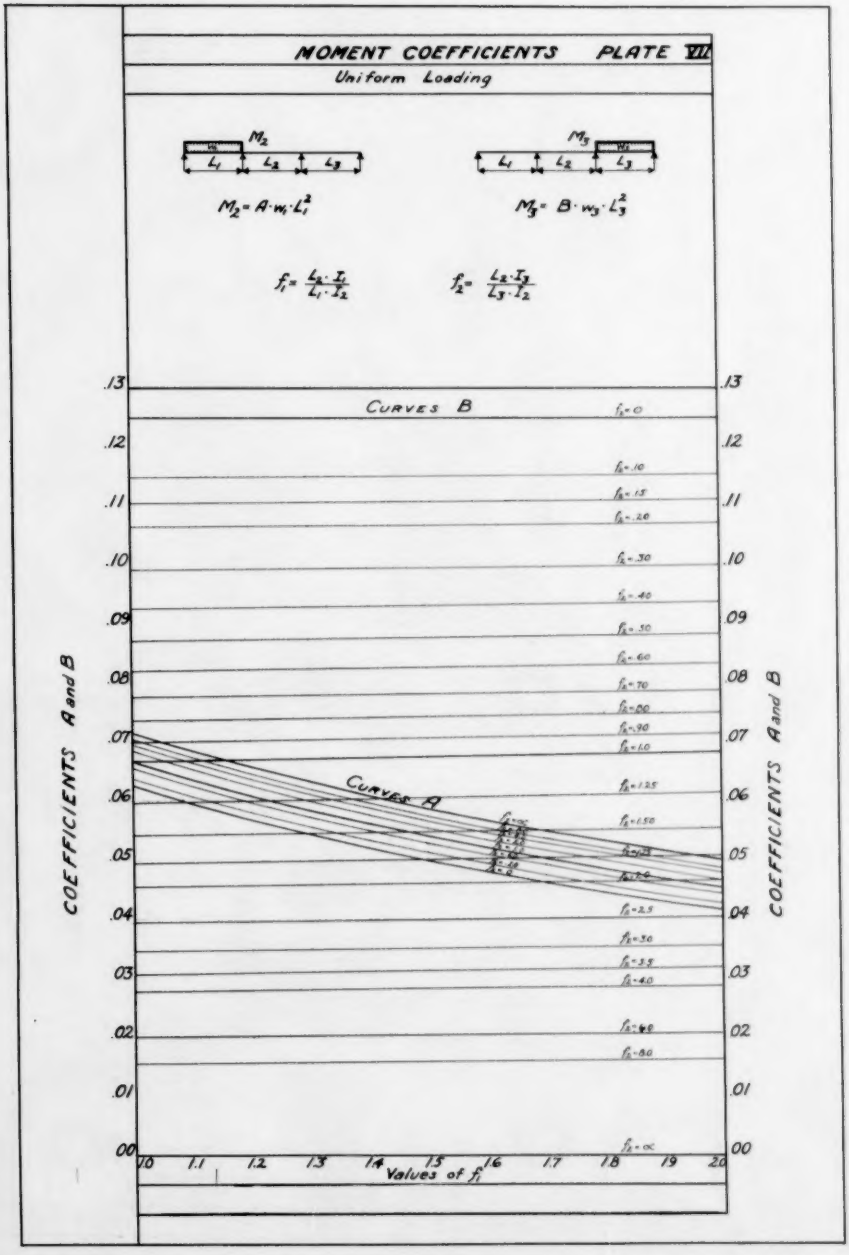
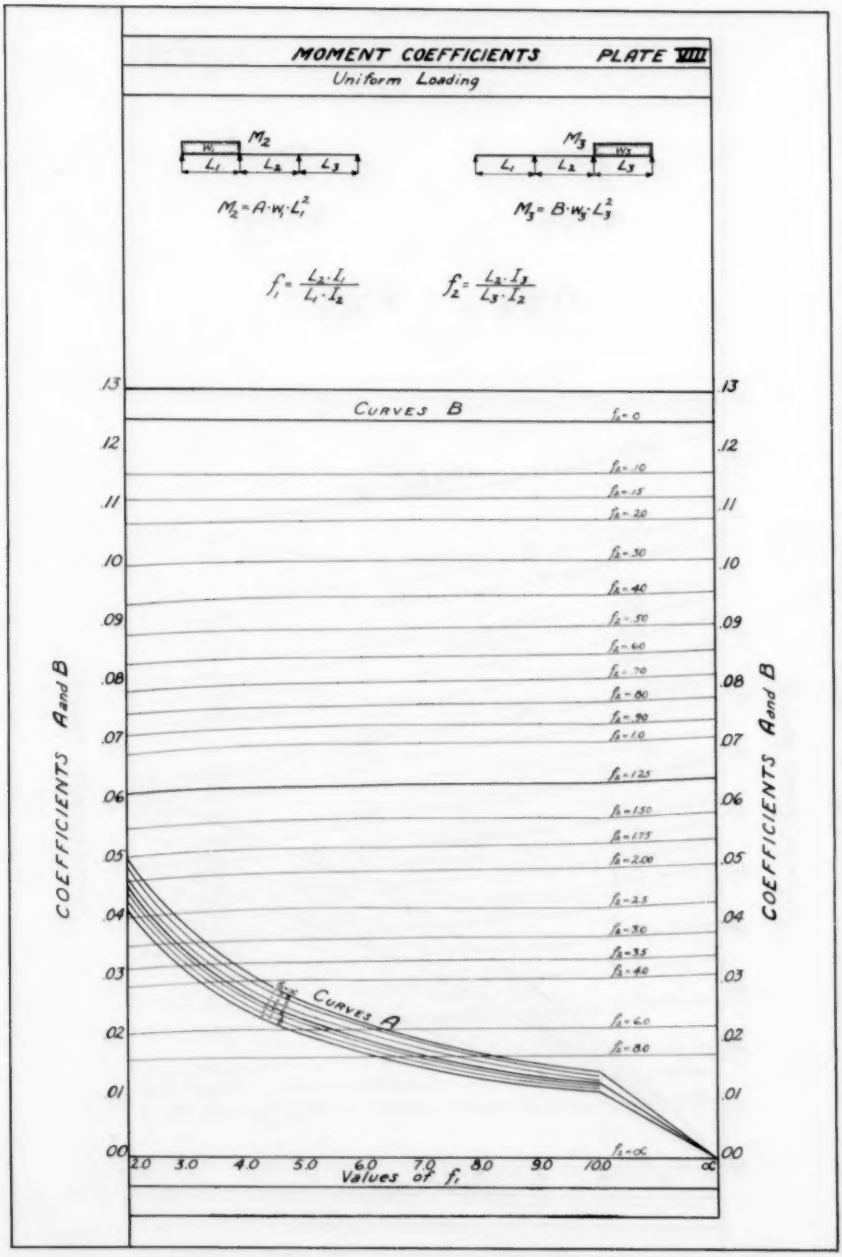
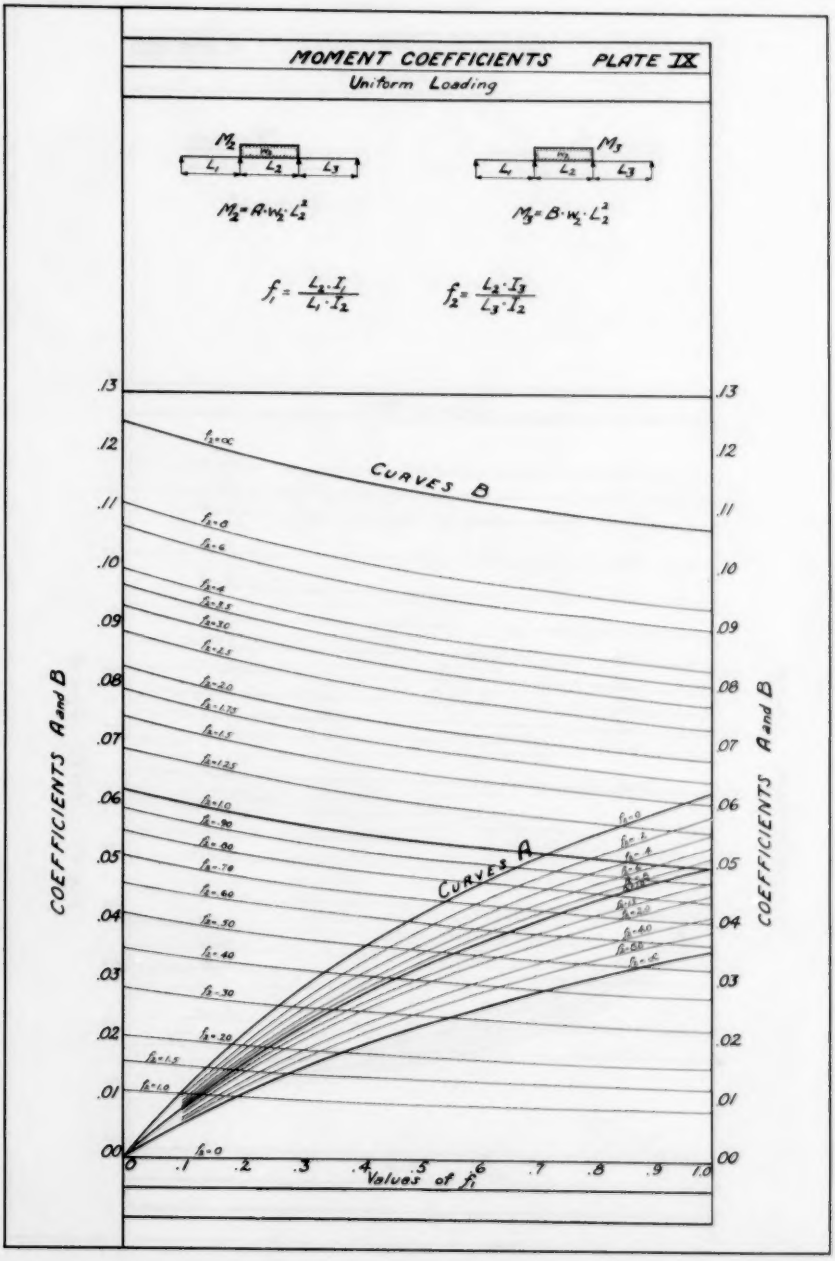
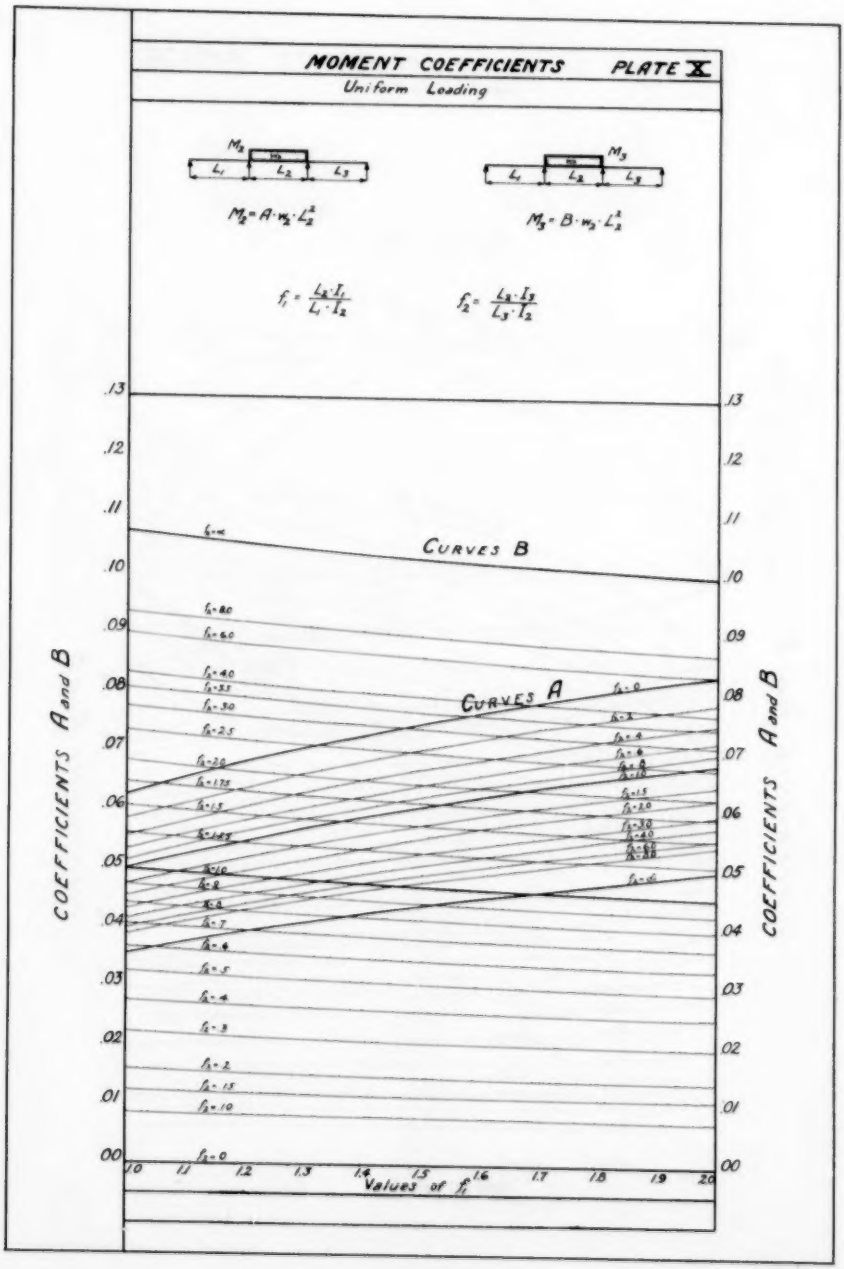
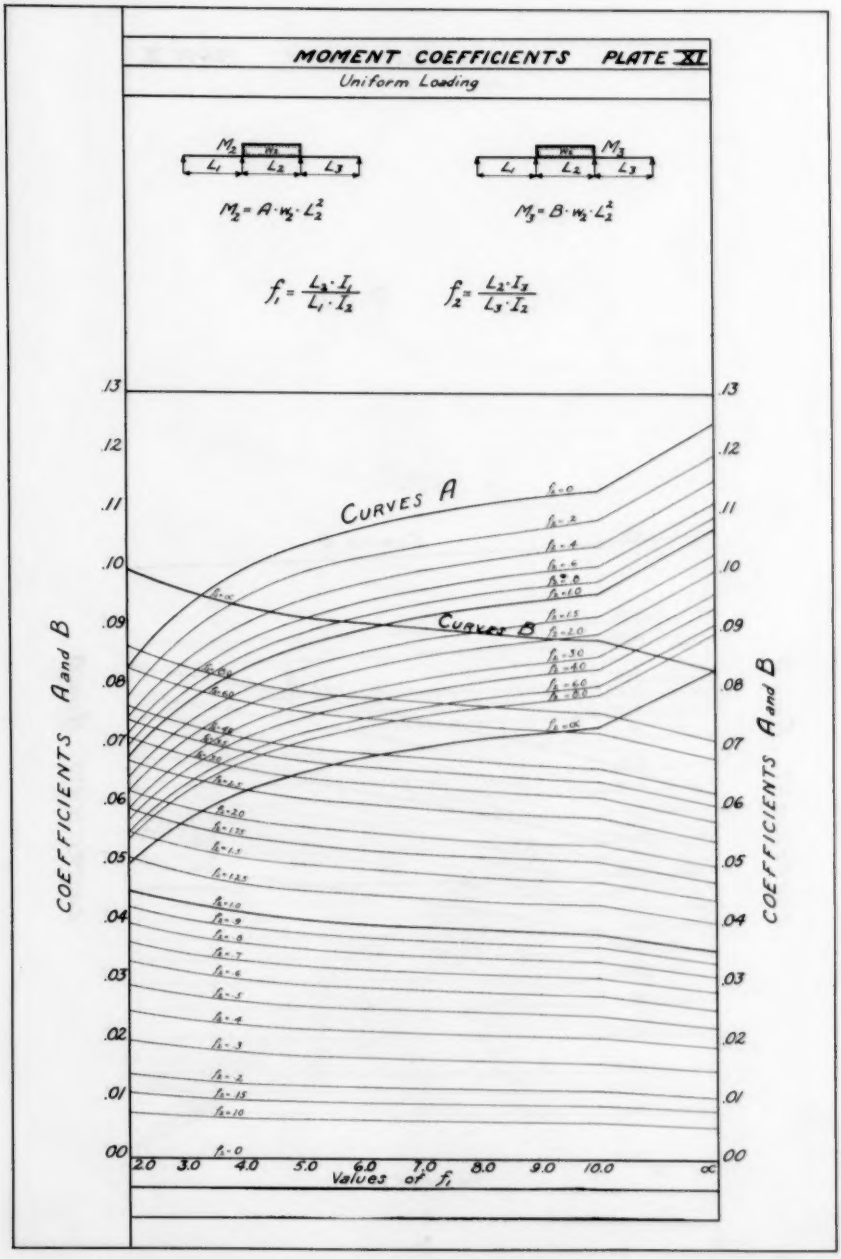

MOMENT COEFFICIENTS PLATE VI
Uniform Loading
M_2
L_1 L_2 L_3
$M_2 = A \cdot w_1 \cdot L_1^2$
M_3
L_1 L_2 L_3
$M_3 = B \cdot w_3 \cdot L_3^2$
$f_1 = \frac{L_2 I_1}{L_1 I_2}$ $f_2 = \frac{L_2 I_3}{L_3 I_2}$
CURVES B
CURVES A
COEFFICIENTS A and B
COEFFICIENTS A and B
.13
.12
.11
.10
.09
.08
.07
.06
.05
.04
.03
.02
.01
.00
$f_1 = 0$
$f_1 = .1$
$f_1 = .15$
$f_1 = .20$
$f_1 = .30$
$f_1 = .40$
$f_1 = .50$
$f_1 = .60$
$f_1 = .70$
$f_1 = .80$
$f_1 = .90$
$f_1 = 1.0$
$f_1 = 1.25$
$f_1 = 1.50$
$f_1 = 1.75$
$f_1 = 2.0$
$f_1 = 2.5$
$f_1 = 3.0$
$f_1 = 3.5$
$f_1 = 4.0$
$f_1 = 6.0$
$f_1 = 8.0$
$f_1 = \infty$
0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
Values of f_1

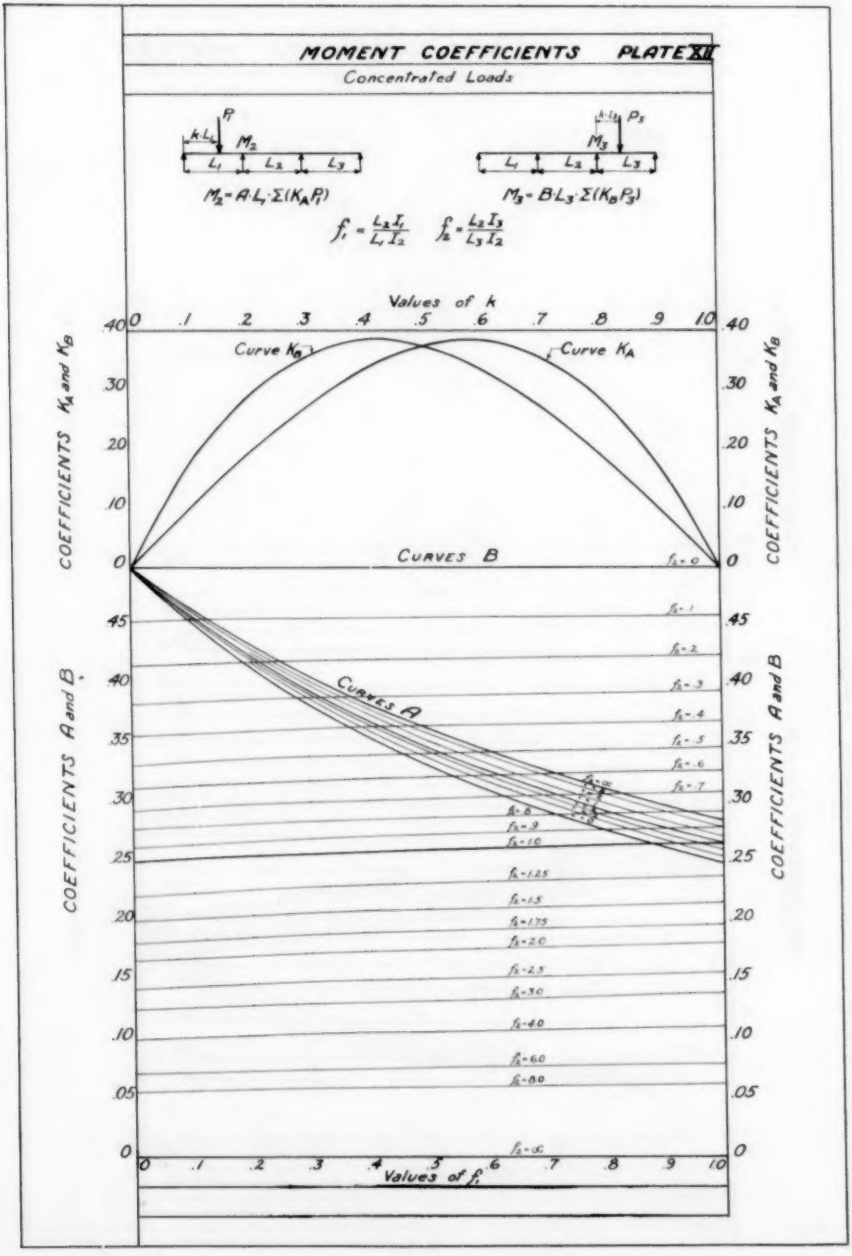

MOMENT COEFFICIENTS PLATE VII
Uniform Loading
M_2
L_1 L_2 L_3
$M_2 = A \cdot w_1 \cdot L_1^2$
M_3
L_1 L_2 L_3
$M_3 = B \cdot w_3 \cdot L_3^2$
$f_1 = \frac{L_2 \cdot I_1}{L_1 \cdot I_2}$
$f_2 = \frac{L_2 \cdot I_3}{L_3 \cdot I_2}$
CURVES B
CURVES A
COEFFICIENTS A and B
COEFFICIENTS A and B
.13
.12
.11
.10
.09
.08
.07
.06
.05
.04
.03
.02
.01
.00
$f_2 = 0$
$f_2 = .10$
$f_2 = .15$
$f_2 = .20$
$f_2 = .30$
$f_2 = .40$
$f_2 = .50$
$f_2 = .60$
$f_2 = .70$
$f_2 = .80$
$f_2 = .90$
$f_2 = 1.0$
$f_2 = 1.25$
$f_2 = 1.50$
$f_2 = 2.0$
$f_2 = 2.5$
$f_2 = 3.0$
$f_2 = 3.5$
$f_2 = 4.0$
$f_2 = 6.0$
$f_2 = 8.0$
$f_2 = \infty$
1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2.0
Values of f_1

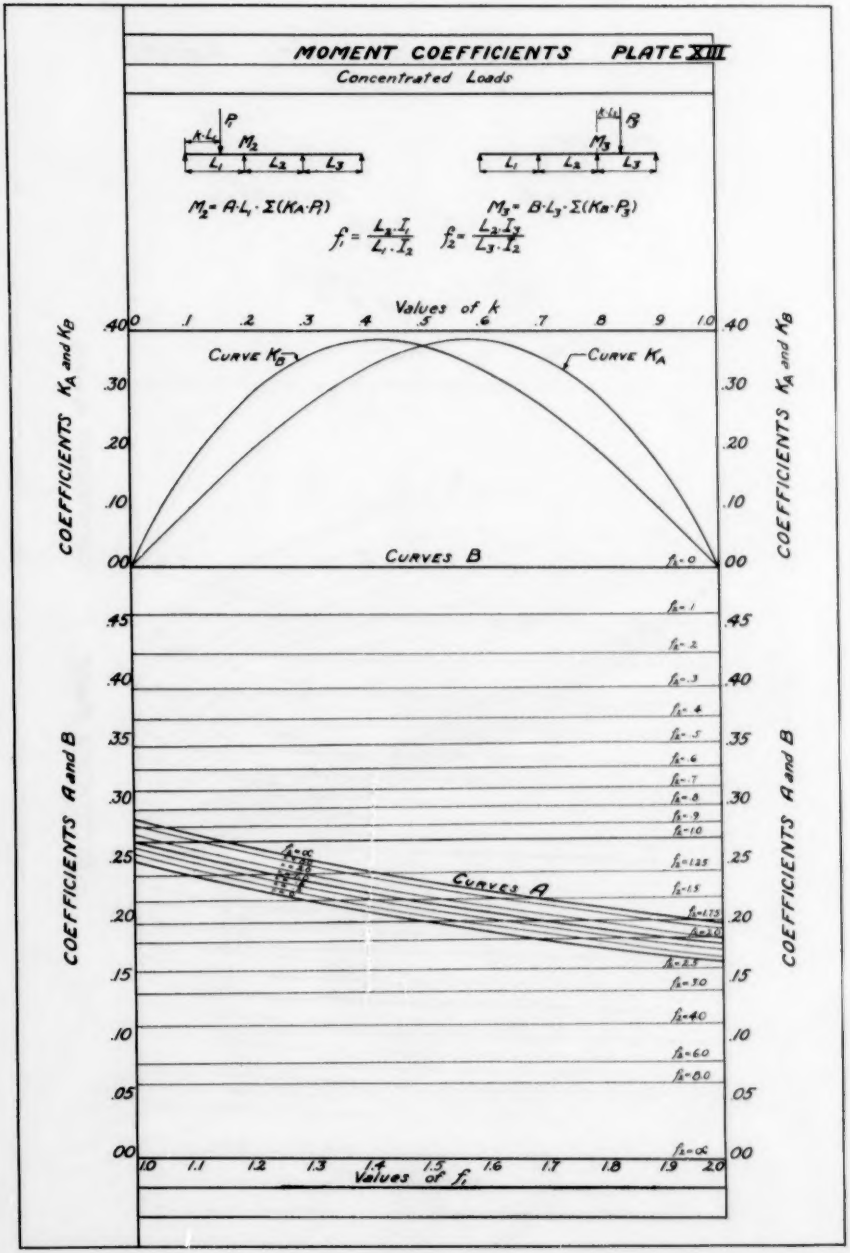

MOMENT COEFFICIENTS PLATE VIII
Uniform Loading
M_2
W_1
L_1 L_2 L_3
$M_2 = A \cdot W_1 \cdot L_1^2$
$f_1 = \frac{L_2 \cdot I_1}{L_1 \cdot I_2}$
M_3
W_3
L_1 L_2 L_3
$M_3 = B \cdot W_3 \cdot L_3^2$
$f_2 = \frac{L_2 \cdot I_3}{L_3 \cdot I_2}$
CURVES B
$f_2 = 0$
$f_2 = .10$
$f_2 = .15$
$f_2 = .20$
$f_2 = .30$
$f_2 = .40$
$f_2 = .50$
$f_2 = .60$
$f_2 = .70$
$f_2 = .80$
$f_2 = .90$
$f_2 = 1.0$
$f_2 = 1.25$
$f_2 = 1.50$
$f_2 = 1.75$
$f_2 = 2.00$
$f_2 = 2.5$
$f_2 = 3.0$
$f_2 = 3.5$
$f_2 = 4.0$
$f_2 = 6.0$
$f_2 = 8.0$
$f_2 = \infty$
CURVES A
COEFFICIENTS A and B
.13 .12 .11 .10 .09 .08 .07 .06 .05 .04 .03 .02 .01 .00
2.0 3.0 4.0 5.0 6.0 7.0 8.0 9.0 10.0 ∞
Values of f_1

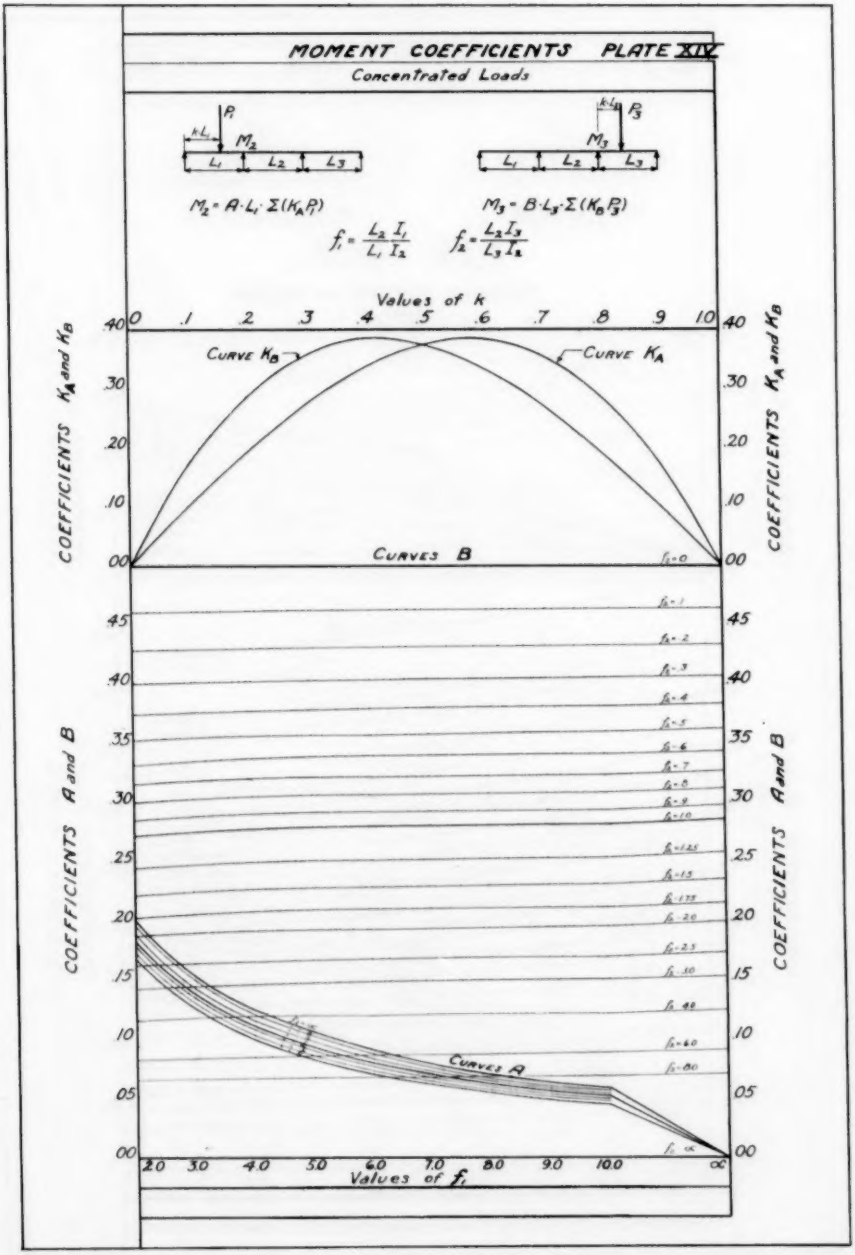

MOMENT COEFFICIENTS PLATE IX
Uniform Loading
M2
L1
L2
L3
M2 = A·W2·L2²
M3
M3 = B·W2·L2²
f1 = L2·I1 / L1·I2
f2 = L2·I3 / L3·I2
CURVES B
CURVES A
COEFFICIENTS A and B
Values of f1
0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
.00 .01 .02 .03 .04 .05 .06 .07 .08 .09 .10 .11 .12 .13

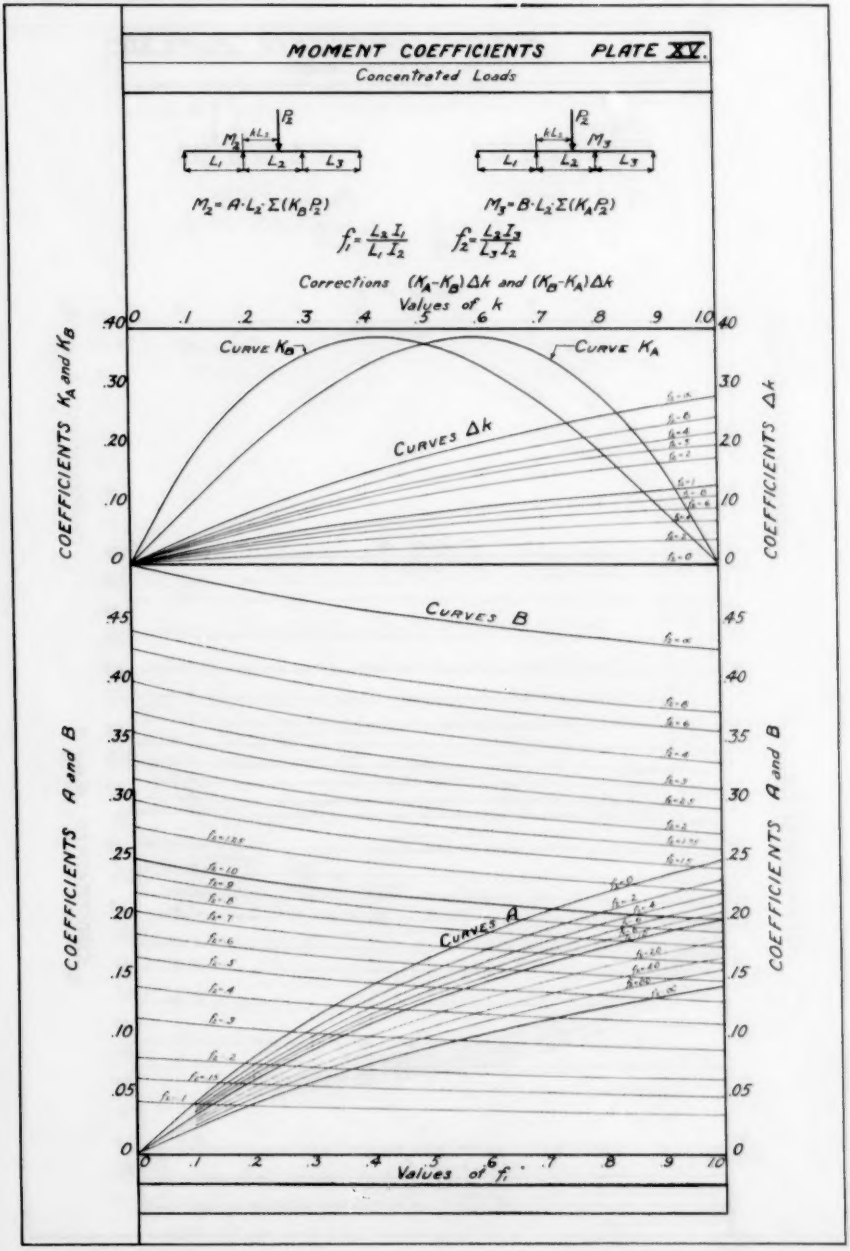

MOMENT COEFFICIENTS PLATE X
Uniform Loading
$M_2 = A \cdot w_2 \cdot L_2^2$
$M_3 = B \cdot w_2 \cdot L_2^2$
$f_1 = \frac{L_2 \cdot I_1}{L_1 \cdot I_2}$
$f_2 = \frac{L_3 \cdot I_3}{L_3 \cdot I_2}$
CURVES B
CURVES A
COEFFICIENTS A and B
Values of f_1
.13
.12
.11
.10
.09
.08
.07
.06
.05
.04
.03
.02
.01
.00
1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2.0

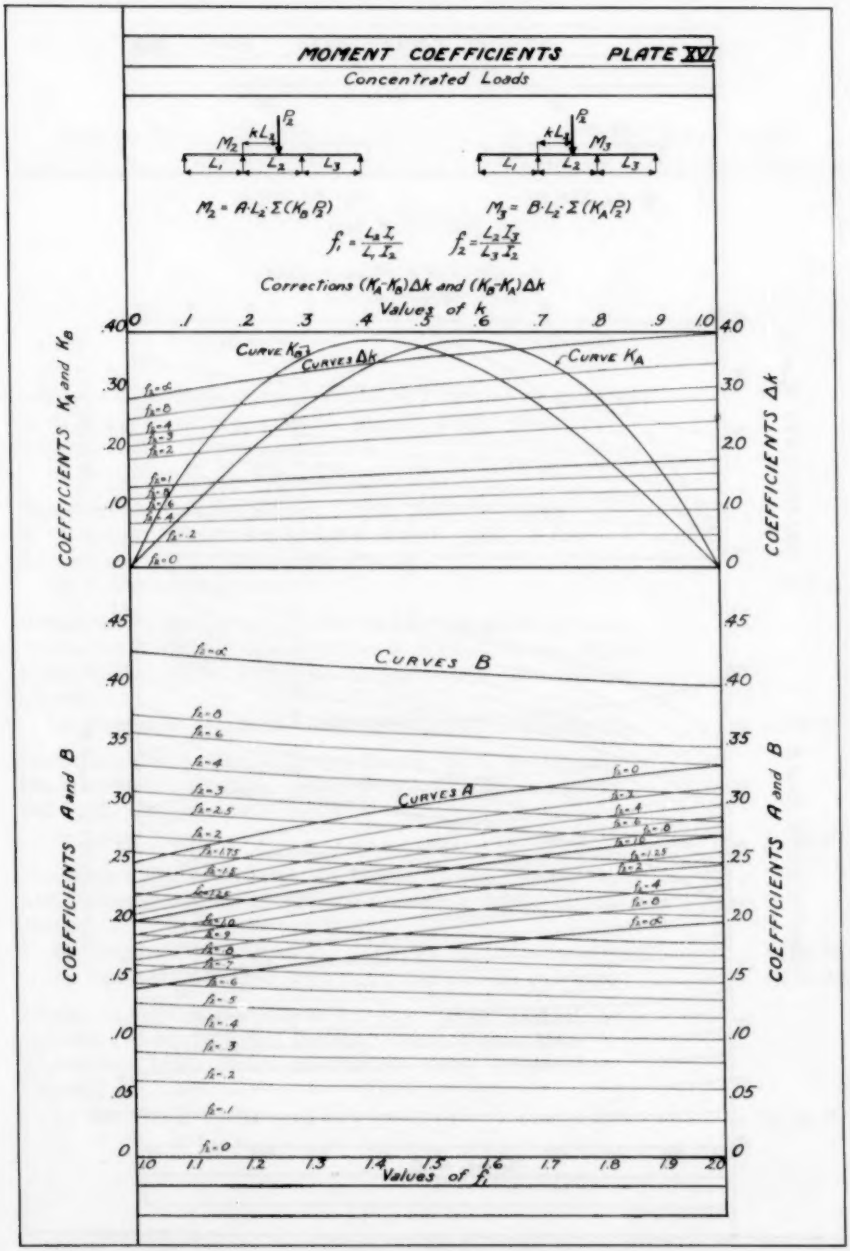

MOMENT COEFFICIENTS PLATE XI
Uniform Loading
$M_2 = A \cdot w_2 \cdot L_2^2$
$M_3 = B \cdot w_2 \cdot L_2^2$
$f_1 = \frac{L_2 \cdot I_1}{L_1 \cdot I_2}$
$f_2 = \frac{L_2 \cdot I_3}{L_3 \cdot I_2}$
CURVES A
CURVES B
COEFFICIENTS A and B
Values of f_1
2.0 3.0 4.0 5.0 6.0 7.0 8.0 9.0 10.0 ∞
.13 .12 .11 .10 .09 .08 .07 .06 .05 .04 .03 .02 .01 .00

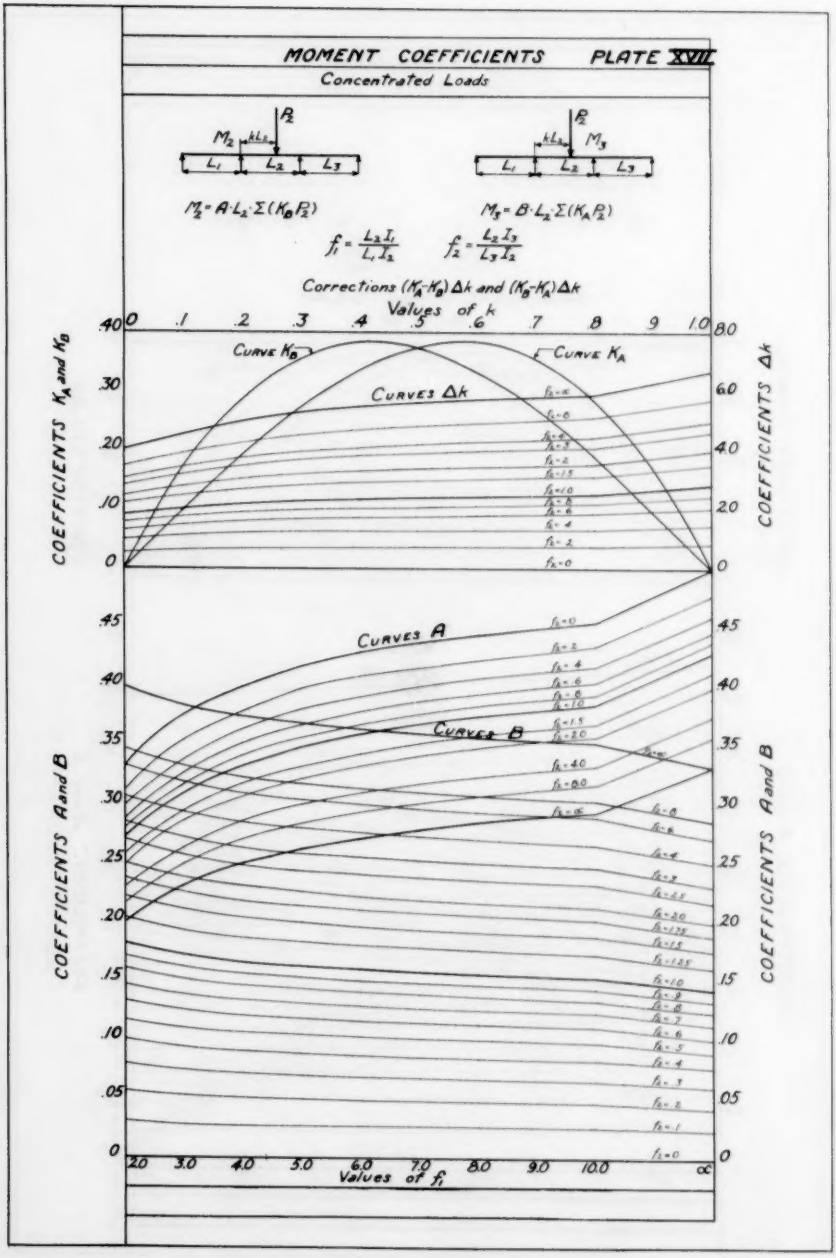

MOMENT COEFFICIENTS PLATE XII
Concentrated Loads
$M_2 = A \cdot L_1 \cdot \Sigma(K_A P_1)$
$M_3 = B \cdot L_3 \cdot \Sigma(K_B P_3)$
$f_1 = \frac{L_2 I_1}{L_1 I_2}$ $f_2 = \frac{L_2 I_3}{L_3 I_2}$
Values of k
Curve K_B
Curve K_A
CURVES B
CURVES A
COEFFICIENTS K_A and K_B
COEFFICIENTS A and B
Values of f_1
f_2 = 0
f_2 = .1
f_2 = .2
f_2 = .3
f_2 = .4
f_2 = .5
f_2 = .6
f_2 = .7
f_2 = .8
f_2 = .9
f_2 = 1.0
f_2 = 1.25
f_2 = 1.5
f_2 = 1.75
f_2 = 2.0
f_2 = 2.5
f_2 = 3.0
f_2 = 4.0
f_2 = 6.0
f_2 = 8.0
f_2 = ∞


MOMENT COEFFICIENTS PLATE XIII
Concentrated Loads
P
k·L1
M2
L1 L2 L3
k·L3
P3
M3
M2 = A·L1·Σ(KA·P)
M3 = B·L3·Σ(KB·P3)
f1 = L2·I1 / L1·I2
f2 = L2·I3 / L3·I2
Values of k
CURVE KB
CURVE KA
CURVES B
f2=0
f2=1
f2=2
f2=3
f2=4
f2=5
f2=6
f2=7
f2=8
f2=9
f2=10
f2=12.5
f2=15
f2=17.5
f2=20
f2=25
f2=30
f2=40
f2=60
f2=80
f2=∞
CURVES A
COEFFICIENTS KA and KB
COEFFICIENTS A and B
Values of f1


MOMENT COEFFICIENTS PLATE XIV
Concentrated Loads
$M_2 = A \cdot L_1 \cdot \Sigma(K_A P_1)$
$M_3 = B \cdot L_3 \cdot \Sigma(K_B P_3)$
$f_1 = \frac{L_2 I_1}{L_1 I_2}$ $f_2 = \frac{L_2 I_3}{L_3 I_2}$
Values of k
CURVE K_B
CURVE K_A
CURVES B
CURVES A
COEFFICIENTS K_A and K_B
COEFFICIENTS A and B
Values of f_1


MOMENT COEFFICIENTS PLATE XV.
Concentrated Loads
$M_2 = A \cdot L_2 \cdot \Sigma(K_B P_2)$
$M_3 = B \cdot L_2 \cdot \Sigma(K_A P_2)$
$f_1 = \frac{L_2 I_1}{L_1 I_2}$ $f_2 = \frac{L_2 I_3}{L_3 I_2}$
Corrections $(K_A - K_B)\Delta k$ and $(K_B - K_A)\Delta k$
Values of k
Curve K_B
Curve K_A
Curves Δk
Curves B
Curves A
COEFFICIENTS K_A and K_B
COEFFICIENTS Δk
COEFFICIENTS A and B
Values of f_1


MOMENT COEFFICIENTS PLATE XVI
Concentrated Loads
$M_2 = A \cdot L_2 \cdot \Sigma (K_B P_2)$
$M_3 = B \cdot L_2 \cdot \Sigma (K_A P_2)$
$f_1 = \frac{L_2 I_1}{L_1 I_2}$ $f_2 = \frac{L_2 I_3}{L_3 I_2}$
Corrections $(K_A - K_B)\Delta k$ and $(K_B - K_A)\Delta k$
Values of k
Curve K_B
Curves Δk
Curve K_A
COEFFICIENTS K_A and K_B
COEFFICIENTS Δk
Curves B
Curves A
COEFFICIENTS A and B
Values of f_1


MOMENT COEFFICIENTS PLATE XVII
Concentrated Loads
$M_2 = A \cdot L_2 \cdot \Sigma(K_B P_2)$
$M_3 = B \cdot L_2 \cdot \Sigma(K_A P_2)$
$f_1 = \frac{L_2 I_1}{L_1 I_2}$ $f_2 = \frac{L_2 I_3}{L_3 I_2}$
Corrections $(K_A - K_B)\Delta k$ and $(K_B - K_A)\Delta k$
Values of k
Curve K_B
Curve K_A
Curves Δk
Curves A
Curve B
COEFFICIENTS K_A and K_B
COEFFICIENTS Δk
COEFFICIENTS A and B
Values of f_1

Journal of the
STRUCTURAL DIVISION
Proceedings of the American Society of Civil Engineers

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ANALYSIS OF MULTIPLE-SPAN CONTINUOUS TRUSSES[a]

Closure by Benjamin C. F. Wei

BENJAMIN C. F. WEI,[1] J.M. ASCE.—It certainly gives the writer great pleasure to read the discussion by Mr. Mullins. Judging from the lengthy and systematic presentation of Mr. Mullins' discussion, one can only conclude that his is a profound interest in the problem of multiple-span continuous trusses. The writer is sincerely thankful for his time and effort.

Essentially, Mr. Mullins has attempted, in his discussion, to reveal the advantages of the writer's procedure of using moment influence lines over the more general procedure of employing reaction redundants in the solution of multiple-span continuous trusses. In order to convince the readers by facts and figures, Mr. Mullins went to great length in presenting, for a four-span unsymmetrical continuous truss, two complete sets of calculations, one for the moment influence lines and the other for the reaction influence lines. By a simple comparison of the works shown by Mr. Mullins in Tables I, II and III against those in Tables IV, V and VI, there should be little doubt that much less computation is involved in the solution for the moment influence lines. In the last paragraph, Mr. Mullins concluded that, for that particular problem, about three times more work would be involved in the solution of reaction influence lines than those of the moments.

Mr. Mullins used a fine approach of least work method in deriving his fundamental Eqs. (4), (5) and (6). Then he proceeded to solve these equations and arrived at the expressions for X_a, X_b and X_c in Eqs. (7), (8) and (9). The writer presumes that Eqs. (4), (5) and (6) were solved in general form in order that, for demonstration purposes, the expressions (7), (8) and (9) would be applicable for solutions of both the moment and reaction redundants. In actual practice, if one is primarily interested in solving a specific problem by using the writer's procedure, it is much simpler to solve the simultaneous equations by substituting beforehand some zero constants. This was done as shown in the solution of Eqs. (7), (8) and (9) in the original paper. There, one zero constant ($\theta_{ac} = \theta_{ca} = 0$) was present. However, as the number of span increases, the number of these zero constants also increases. In general, if the constants for the simultaneous equations are written in matrix form, the matrix would require, for reaction influence lines, (n - 1) number of rows and columns (n being the number of spans) whereas, for moment influence lines, it would require the same number of columns but not more than three constants in each row regardless the number of continuous spans. This point, incidentally, was well brought out by Mr. Chang in his discussion.

Aside from using a number of additional symbols for simplifying the

a. Proc. Paper 1187, March, 1957, by Benjamin C. F. Wei.

1. Structural Designer, D. B. Steinman, Cons. Engr., New York, N. Y.

appearances of the simultaneous equations and the headings of the tables, Mr. Mullins' computation of moment influence lines is essentially similar to the one given in the paper. In the moment and reaction influence line computations Mr. Mullins uses, respectively, five columns in Table I and six columns in Table IV in evaluating the constants for use in the equations. It is difficult to understand why Mr. Mullins prefers to compute these constants by laborious multiplications and by addition of a long string of numbers rather than to use the writer's method of end shear, obtained by simply adding a few figures together. Certainly, for purpose of simplification, the five columns (7, 10, 13, 14 and 15) in Table I and six columns (7, 10, 13, 14, 15 and 16) in Table IV in Mr. Mullins' computation sheets may be eliminated. Hence, Tables I and IV would be reduced to a form similar to the one given by the writer in the paper. As for the checking of calculations, in view of the magnitude of work usually involved in the analysis of a long-span continuous structure, the writer would consider it a good office practice to assign, in the final design, a separate junior designer or a computer to give an immediate check of all the figures worked out by the designer to insure that all the ensuing computations would be free from errors that may be carried over from the earlier computations.

The writer is grateful to Mr. Mullins in pointing out the suitability of writer's procedure to computation by the use of a slide rule. As pointed out in the discussion, the results of moment influence lines as given by Mr. Mullins, using a slide rule, agree surprisingly well with those given by the writer, using a desk calculator. The writer agrees heartily to the statement by Mr. Mullins that 'if one selects the reactions as the redundants, it is necessary to deal with much longer lever arms and longer stresses and consequently the effect of small errors or approximations contribute to errors in the final result.'

Mr. Mullins has made a valuable supplement to the paper by showing in his discussion a solid comparison of two methods of computing influence lines for a four-span unsymmetrical continuous truss. By so doing, he has affirmed beyond doubt the writer's assertion of the simplicity of his procedure in solving multiple-span continuous trusses. The writer wishes to thank Mr. Mullins for his tremendous efforts in preparing this valuable discussion which has certainly enriched the contents of the paper.

The comments by Dr. Chang are of great interest. The writer is pleased to find that, due to the repetitive nature, his procedure is well suited to programming on an electronic digital computer. With the increasing use of these high speed digital computers in civil engineering problems and with the trend that more design offices are being equipped with the budget-priced computers such as the Bendix G15-D, Royal McBee LGP 30 and others, it is not hard to foresee that in the near future continuous truss analysis may be made by these machine computations. As pointed out by Dr. Chang, in order to adapt the procedure to machine operations, the simplest method is to arrange the equations in matrix form and perform, by the use of appropriate codes, the inverse and multiplication of the matrices by the computer. In a paper[2] presented at ASCE Annual Convention in October, 1957, it was reported that analyses for structures having as high as 106 independent coordinates have been performed on IBM 701 computer using the technic of matrices.

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2. Archer, J. S., "The Stiffness Matrix Method of Structural Analysis as Used on High Speed Digital Computers," ASCE Proc. Paper (To Be Published).

In the event that the program for the analysis of continuous trusses should be available, two immediate effects may be forecasted; (1) that analysis for long-span trusses with four, five or many more spans would not be considered laborious and cumbersome and the design of these long-span bridges would be entirely within reach of the designing engineers, and (2) that more accurate preliminary studies for a long-span continuous truss may be made in determining an economical layout of the outline of the truss without sacrificing the limited design time which is often a pressing factor in design offices.

It must be noted, however, that the initial setting-up and the coding of a specific problem are difficult and expensive. For a single structure, it may not pay for an office to initiate a computer program. On the other hand, for a long crossing where there is a repetition of similar structures, it is reasonable to believe that faster and better design could be achieved by the use of computers. Furthermore, attention must be called to the fact that the computer program set up for the analysis can not only be used to find the primary stresses in the structure but also can be conveniently used to find many other types of stresses required in the course of complete design of a continuous bridge, such as the stresses due to differential settlement or construction inaccuracy in elevation of bridge piers, temperature stresses due to restraint of supports, erection stresses during many stages of cantilever erections, etc.

The writer thanks Dr. Chang for his excellent remarks, commenting on the procedure in the light of modern computation method using high-speed digital computers.

After the paper was published, the writer received some queries about the role of verticals in affecting the computation of moment influence lines in a continuous truss. To clarify this problem, it may help, perhaps, to review briefly the fundamentals of the derivation of elastic weight in a trussed structure. In Fig. 1 is shown a simple Pratt truss. The derivation of elastic weights by the method of influence line may be summarized in the following four steps:

(1) By method of least work, $1 \times \delta = u \times \Delta\ell$,
(2) Therefore, influence line for stress (u) in member a - b x $(\Delta\ell)$ is equal to the deflection curve for the truss due to a change of length of member a - b,
(3) By principle of moment area method, elastic weights are equivalent to loads, the moment curve of which corresponds to deflection curve, and
(4) It follows, then, that the elastic weights are equal to loads, the moment curve of which is such as to correspond to the stress influence line of member a - b x $(\Delta\ell)$.

By following the above steps, one can obtain the elastic weights due to the changes of length of two typical verticals in the Pratt truss as shown in Fig. 1. It must be noted that for odd panel verticals in the truss of Fig. 1, such as the vertical e - f, and for Warren type of truss as shown in the illustrated example of the paper, $\Delta\ell$ due to unit moment = 0; hence, elastic weights for these members are equal to zero.

In closing, it may be concluded that the writer's procedure of using moment influence lines in analyzing multiple-span continuous trusses meets the designer's demand of a "practical method of analysis that is accurate, systematic, easy to work and yet simple enough to be plainly understandable in its principle of application." The procedure is highly adaptable to use in design

offices, as it has been successfully applied by the writer in the design of numerous continuous trusses in the Kingston Bridge and the Mackinac Straits Bridge. Finally, the writer wishes to thank again Messrs. Mullins and Chang for their enlightening and valuable discussions.

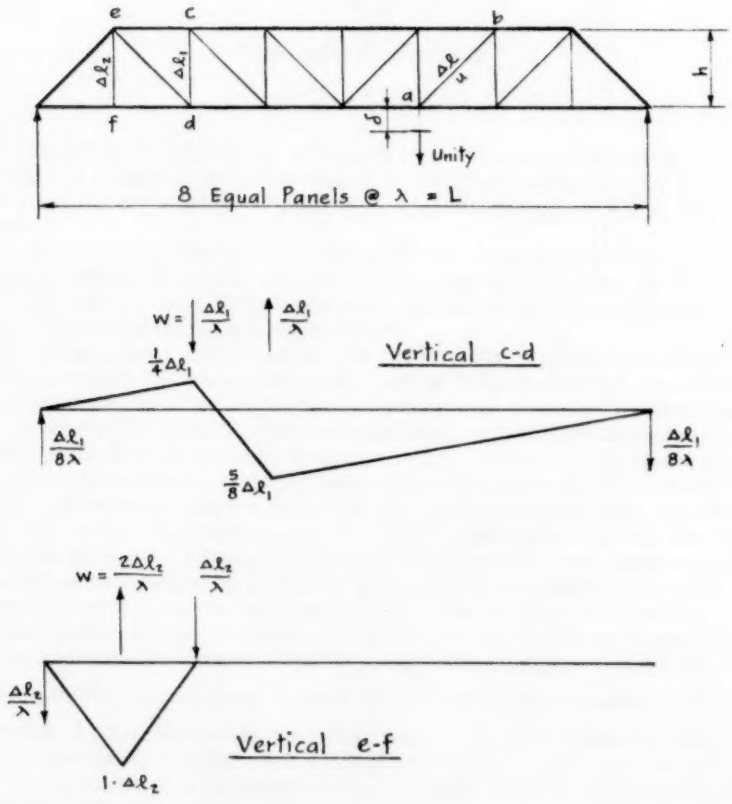


Fig. 1 Elastic Weights Due To Change of Length of Verticals in a Pratt Truss

RESISTANCE TO OVERTURNING OF SINGLE, SHORT PILES[a]

Closure by E. Czerniak

E. CZERNIAK,[1] A.M. ASCE.—The writer would like to express his appreciation to Messrs. A. O. Grote and J. V. Du Plessis for the interest shown in the subject, and to thank them for the interesting points raised in their discussion.

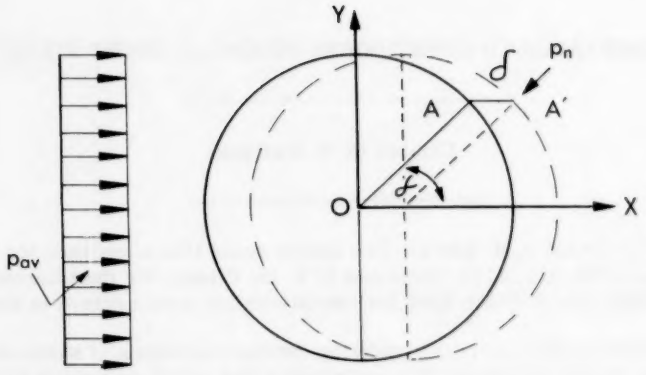
Mr. Grote is quite correct in pointing out the limitations of some of the "accepted" design formulas. His contribution has added materially to the paper, and with his well chosen examples, Mr. Grote pin-pointed some of the areas in which clarification was necessary.

The writer cannot agree with Mr. Du Plessis that "cylinders could be treated as rectangles having the same width as their diameter." Experience has shown that a member with a curved surface can penetrate the earth easier than one with a flat surface. The question, then, is the magnitude of the numerical value to be attached to the shape factor. Obviously, the shape factor can best be determined from field tests. The overall factor is a variable quantity. It depends upon the shape and size of the cross-section, the adhesion of the soil, physical properties, the magnitude of the applied force, and the spacing of the piles. For a single pile the abutting resistance may be much greater than that caused by a strip of earth equal to the width of the pile, and may be allowed for, in the efficiency factor. Just as there are several methods for calculating the pressure against the pile, so there is also no agreement in the profession, as to the proper value of the shape factor. For a round pile the value commonly used for the maximum pressure is $\frac{\pi}{2}$ or 1.57 times the average on the projected area. This is the same as saying that the effectiveness of the pile with a circular cross-section is equal only to 0.636 as compared to a rectangular cross-section equal in width to the pile diameter. The effectiveness of the round pile is, unquestionably, higher than that. If it is assumed, as before, that the resistance developed by the earth is directly proportional to the compression to which it is subjected, the following analysis may be helpful.

As the pile moves a distance δ the soil at A is compressed an amount $\delta \cos \alpha$. Then, p_n is equal to p cos α in which p is the maximum pressure against the middle element (at $\alpha = 0$ cos $\alpha = 1$). The components of p_n in the Y direction are equal and opposite on either half of the pile, and act to compress the soil laterally, and thus aid in the penetration. The component of p_n

a. Proc. Paper 1188, March, 1957, by E. Czerniak.

1. Design Eng. Supervisor, The Fluor Corp., Ltd., Los Angeles, Calif.



in the X direction equals $p_n \cos \alpha = p \cos^2 \alpha$. The element of resistance equals

$$p_N \cos\alpha \, ds = p \cos^2\alpha \, \frac{D}{2} \, d\alpha$$

$$p_{av} \times D = p \frac{D}{2} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^2\alpha \, d\alpha = p D \frac{\pi}{4}$$

hence

$$p_{av} = \frac{\pi}{4} p = 0.785 p$$

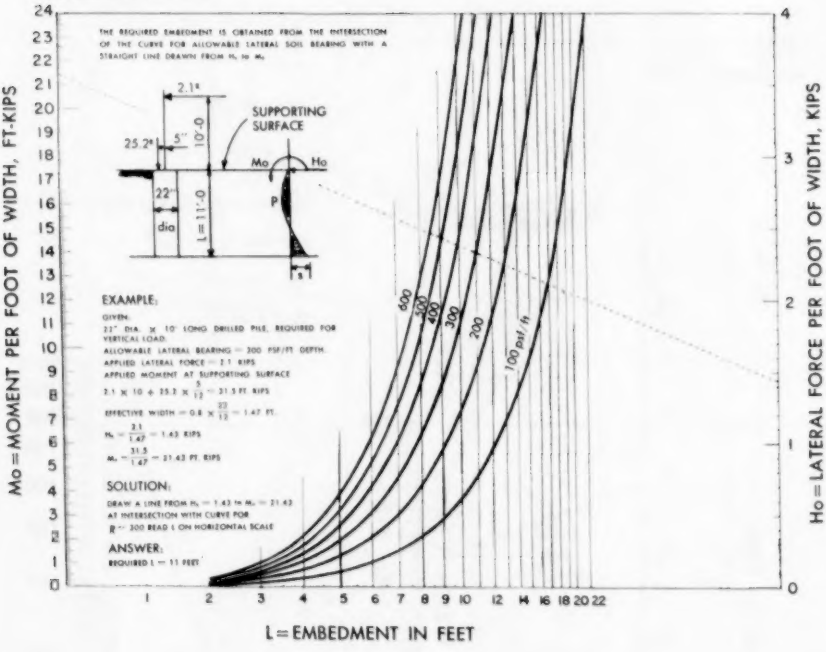
Although the elastic approach may be helpful in providing a rational answer, many other factors must be considered, and a factor of 0.8 is recommended.

Fig. 7, in the paper, was based on the $\frac{2}{\pi}$ factor. It would be convenient to have a chart in which the relationship between the variables are given for the basic flat surface. Fig. 7(a) represents the necessary embedment of a pile with a rectangular cross-section. When using the chart for the required embedment of round piles, the effective width may be taken as 0.8 times the pile diameter.

Similarly, Fig. 11(a) is presented for the required embedment of rectangular piles, restrained against translation. In either case, the embedment to resist lateral loading should be at least 4 feet into natural soils or properly compacted fills.

Mr. Du Plessis favors "in many cases to assume the pressure independent of the deflection." Such an assumption would lead to a pressure diagram of the plastic nature, which would be justified as an ultimate procedure, provided deflections under service loads are not excessive. In such cases the rectangular pressure diagram is by far the simplest. However, as Mr. Grote correctly stated the parabolic diagram approaches most closely known facts. It is true that, without design aids, it is somewhat awkward to use. The purpose of the paper was to provide such aids, and hope of stimulating large scale tests and fruitful discussion among engineers which may lead to the formulation of

FIG. 7(a)


THE REQUIRED EMBEDMENT IS OBTAINED FROM THE INTERSECTION OF THE CURVE FOR ALLOWABLE LATERAL SOIL BEARING WITH A STRAIGHT LINE DRAWN FROM H_o TO M_o
2.1k
SUPPORTING SURFACE
25.2k
5"
10'-0
Mo
Ho
22"
dia
L = 11'-0
P
EXAMPLE:
GIVEN:
22" DIA. X 10' LONG DRILLED PILE, REQUIRED FOR VERTICAL LOAD.
ALLOWABLE LATERAL BEARING = 300 PSF/FT. DEPTH.
APPLIED LATERAL FORCE = 2.1 KIPS
APPLIED MOMENT AT SUPPORTING SURFACE
$2.1 \times 10 + 25.2 \times \frac{5}{12} = 31.5$ FT. KIPS
EFFECTIVE WIDTH $= 0.8 \times \frac{22}{12} = 1.47$ FT.
$H_o = \frac{3.1}{1.47} = 1.43$ KIPS
$M_o = \frac{31.5}{1.47} = 21.43$ FT. KIPS
SOLUTION:
DRAW A LINE FROM $H_o = 1.43$ TO $M_o = 21.43$ AT INTERSECTION WITH CURVE FOR $R = 300$ READ L ON HORIZONTAL SCALE
ANSWER:
REQUIRED L = 11 FEET
600
500
400
300
200
100 psf /ft
Mo = MOMENT PER FOOT OF WIDTH, FT-KIPS
Ho = LATERAL FORCE PER FOOT OF WIDTH, KIPS
L = EMBEDMENT IN FEET

a design criteria. Many of the items discussed are being studied in various research projects now in progress. The results of these investigations may provide some of the answers about the validity of the assumptions and the resulting analytical procedures.

As Mr. Grote points out the maximum shear and moment may be approximated from the Shear and Moment Coefficients of Figures 9 and 10.

Based on the assumed pressure distribution along the rigid pile, coefficients for the location and magnitude of maximum shear and moment were calculated and plotted in Figures 12 and 13, respectively.

It should be emphasized that no matter how intricate and lengthy the calculations may be the results are not any better than the assumptions upon which the analytical procedures are based.

The numerical coefficients were calculated with the aid of an electronic digital computer. With the increasing use of digital computers in structural design a more refined analysis becomes practical. However, until such analyses and the assumptions are verified by field and laboratory experiments the computer's correct solution of the more and more complicated equations provides assurance of mathematical accuracy only. For the present, no great accuracy can as yet be expected from such computations, and the writer would like to close this discussion with the hope, that the case of the lateral resistance of piles, long and short, is far from being "closed".

FIG. 11(a)

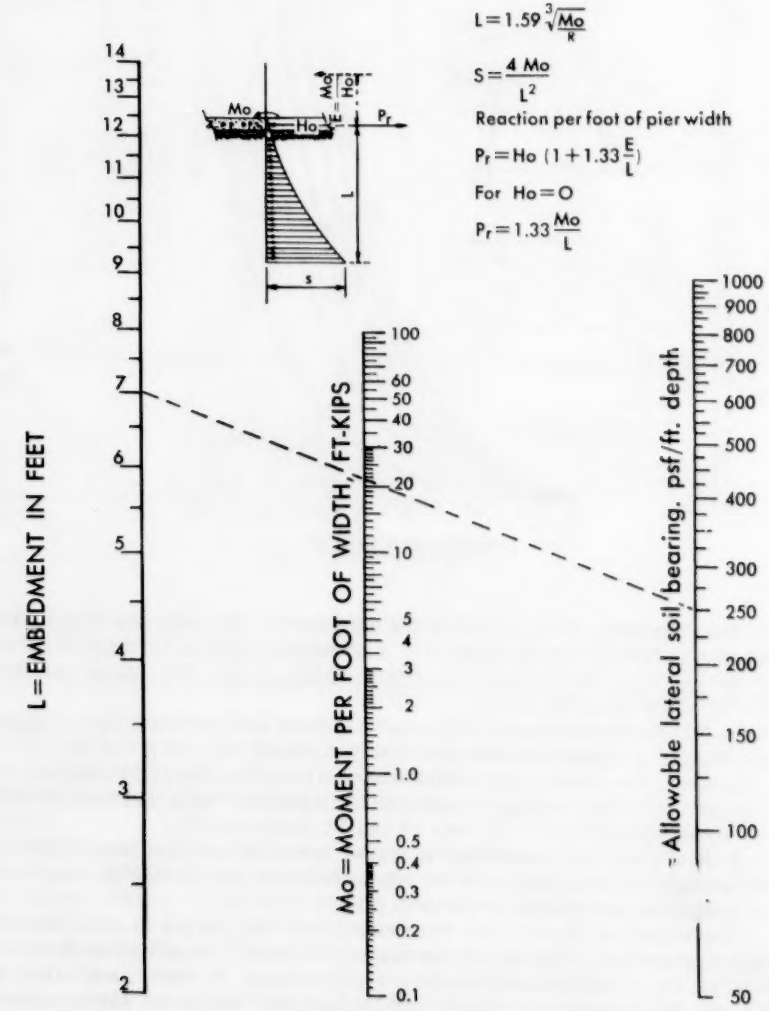
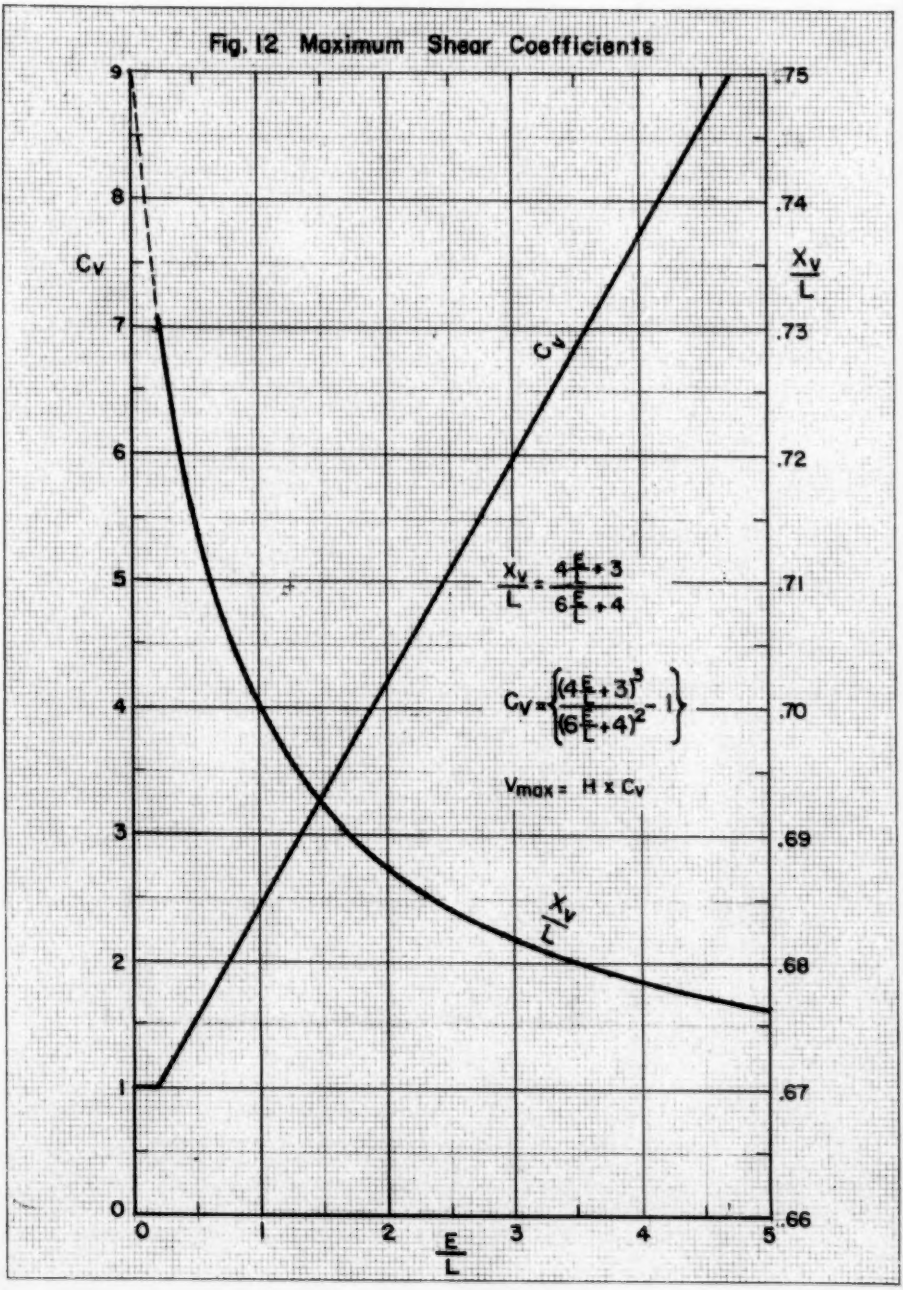

$L = 1.59 \sqrt[3]{\frac{Mo}{K}}$
$S = \frac{4\ Mo}{L^2}$
Reaction per foot of pier width
$P_r = Ho\ (1 + 1.33 \frac{E}{L})$
For Ho = O
$P_r = 1.33 \frac{Mo}{L}$
Mo
Ho
Pr
L
s
L = EMBEDMENT IN FEET
Mo = MOMENT PER FOOT OF WIDTH, FT-KIPS
= Allowable lateral soil bearing. psf/ft. depth

Fig. 12 Maximum Shear Coefficients


C_V
$\frac{X_V}{L}$
C_V
$\frac{X_V}{L}$
$\frac{X_V}{L} = \frac{4\frac{E}{L}+3}{6\frac{E}{L}+4}$
$C_V = \left\{\frac{(4\frac{E}{L}+3)^3}{(6\frac{E}{L}+4)^2} - 1\right\}$
$V_{max} = H \times C_V$
$\frac{E}{L}$
0 1 2 3 4 5
0 1 2 3 4 5 6 7 8 9
.66 .67 .68 .69 .70 .71 .72 .73 .74 .75

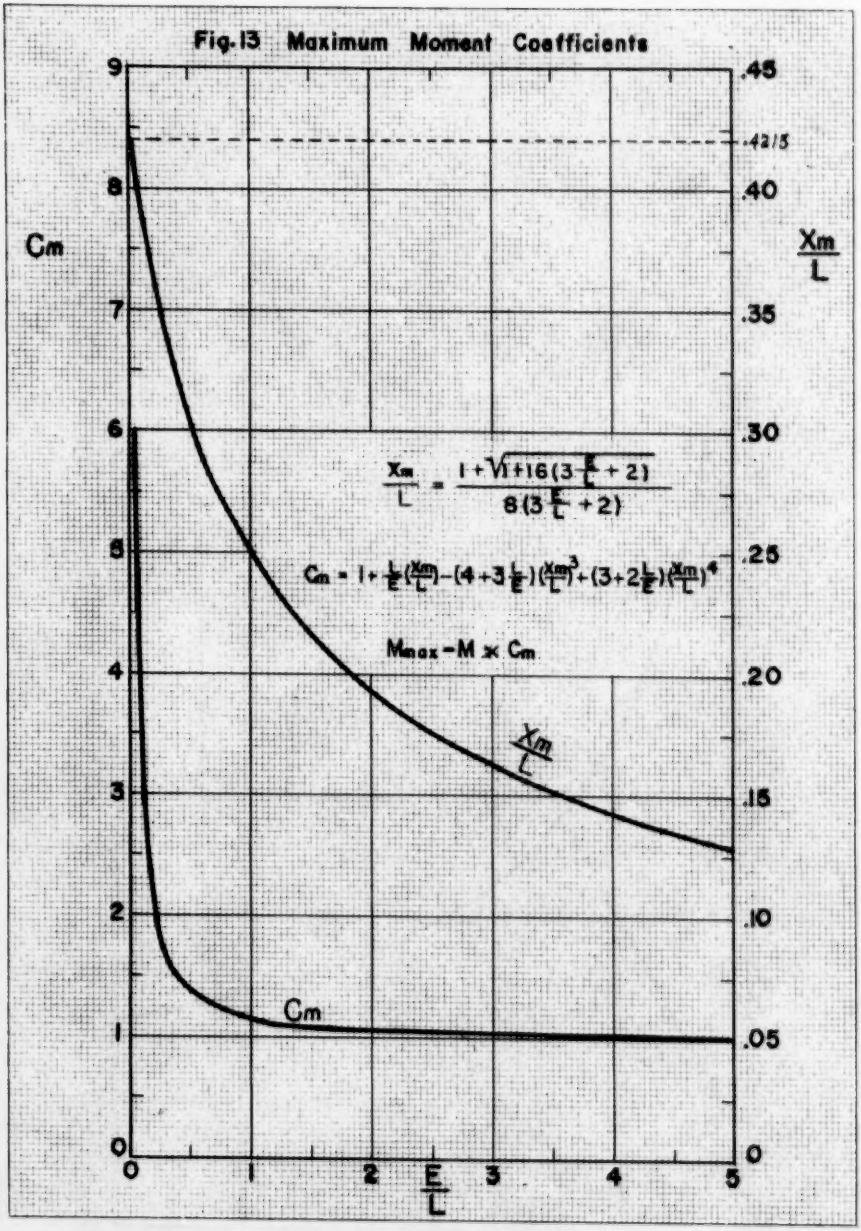


Fig. 13 Maximum Moment Coefficients
Cm
Xm/L
.42/5
$\frac{X_m}{L} = \frac{1+\sqrt{1+16(3\frac{E}{L}+2)}}{8(3\frac{E}{L}+2)}$
$C_m = 1+\frac{1}{E}(\frac{X_m}{L})-(4+3\frac{L}{E})(\frac{X_m}{L})^3+(3+2\frac{L}{E})(\frac{X_m}{L})^4$
$M_{max} = M \times C_m$
Xm/L
Cm
E/L
0 1 2 3 4 5
0 1 2 3 4 5 6 7 8 9
.45 .40 .35 .30 .25 .20 .15 .10 .05 0

DESIGN OF FLEXIBLE STEEL ARCHES BY INTERACTION DIAGRAMS[a]

Closure by Haaren A. Miklofsky and Omar J. Sotillo

HAAREN A. MIKLOFSKY[1] and OMAR J. SOTILLO.[2]—The writers thank Mr. Garcia for his interest in discussing their paper. Mr. Garcia raises three questions that can be satisfactorily answered in a qualitative sense.

1. If the movement of the inflection point for the flexible system with respect to that for the rigid system is considered, the alterations are negligible. Calculations on the Two-Hinged Rainbow arch indicate that L_o increases about 3%, and in the case of the Hingeless Rainbow arch increases about 1%.[3] Using these new values of L_o in the interaction expressions may give a more accurate set of values for m, n and M, but practically the same as the values obtained without consideration of the change in L_o.
2. It is true that the value of H changes as the arch axis deflects. This change in H is less than 1/2%. The error in assuming H constant, however, becomes insignificant once ΔH is multiplied by the value of m. For example, if H = 7500 kips, then ΔH = 38 kips, and for a deflection of one foot, $\Delta M = \Delta Hm = (38)(1) = 38$ ft-kips. This ΔM value can be neglected in a long span structure.
3. The centroid of all downward forces acting to the right of the section under consideration will move relative to the load points as Mr. Garcia implies. Consideration of this problem, however, would involve the determination of the horizontal deflections of all the load points and the location of a new centroid for the downward forces. This would be long and cumbersome, and will not introduce any worthwhile accuracy. Furthermore, to the writers' knowledge, no deflection theory considers such refinements.

The writers were pleased to note that Mr. Garcia finds the paper a useful contribution to structural literature.

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a. Proc. Paper 1190, March, 1957, by Haaren A. Miklofsky and Omar J. Sotillo.

1. Associate Prof. of Civ. Eng., Rensselaer Polytechnic Inst., Troy, N. Y.
2. Cons. Engr., Quinta Mabel, Las Palmas, Caracas, Venezuela, S. A.

3. "Rainbow Arch Bridge over Niagara Gorge: A Symposium," Transactions American Society of Civil Engineers, No. 110, 1945, p. 27 and 32.

DISTRIBUTION OF LOADS ON BRIDGE DECKS

Discussion by Louis Balog

LOUIS BALOG.[1]—Load distribution and moments in two simple span grid bridges were reported in this paper. The 120 ft. span deck plate girder bridge has 6 girders framed by 4 intermediate diaphragms and by a floor slab of unknown thickness resting at the tops of the girders. (Fig. 1). The 70 ft span precast prestressed concrete girder bridge has likewise 6 girders, however, framed by 7 intermediate diaphragms and by a prestressed concrete slab, 7 in thick, anchored to the top of the girders. (Fig. 3).

A simple span grid with $\underline{m}$ girders and $\underline{n}$ cross frames is $n(m - 2)$ times statically indeterminate; considering the torsional rigidity of the girders only, the number of unknowns increases to $n(2m - 2)$, or in the case of 6 girders 2.5 times. The application of the method of load transformation, published by W. Ludwig Andree for rectangular and diagonal grids in 1917,[(1)] often results in halving the above indicated number of the simultaneous elastic equations; this was found also by the author.

120 Ft Span Deck Plate Girder Bridge

In structural arrangement this bridge conforms to the AASHO specifications which require that deck plate girder spans shall have intermediate cross frames at intervals not exceeding 25 ft. The presentation of the moments due to H 20-S 16-44 truck loading by the author is valuable in that comparisons with the results of simple computations and with the AASHO moment requirements can be made. Fig. 1 shows the computation of empirical influence lines of load distribution by a method which was used during the first decade of the century in Hungary.[(2)] Table 1 shows that the evaluation of these influence lines and the computations of the author give identical results, furthermore, that the current (6th Edition) and the proposed (7th Edition) AASHO specifications greatly deviate from these moment values. The proposed AASHO specifications require for the outside girders a moment that is twice as large as the currently specified value and which is identical with that obtained by the author in applying the method of Guyon. Unfortunately, no measured moments were included in the paper for this bridge which would indicate the relative correctness of the moments listed in Table 1.

70 Ft Span Precast Prestressed Concrete Girder Bridge

The properties of this grid indicate straight line load-distribution diagrams significantly rotated by its torsional stiffness. This is suggested also by the

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a. Proc. Paper 1303, July, 1957, by A. M. Lount.

1. Cons. Engr., Binghamton, N. Y.

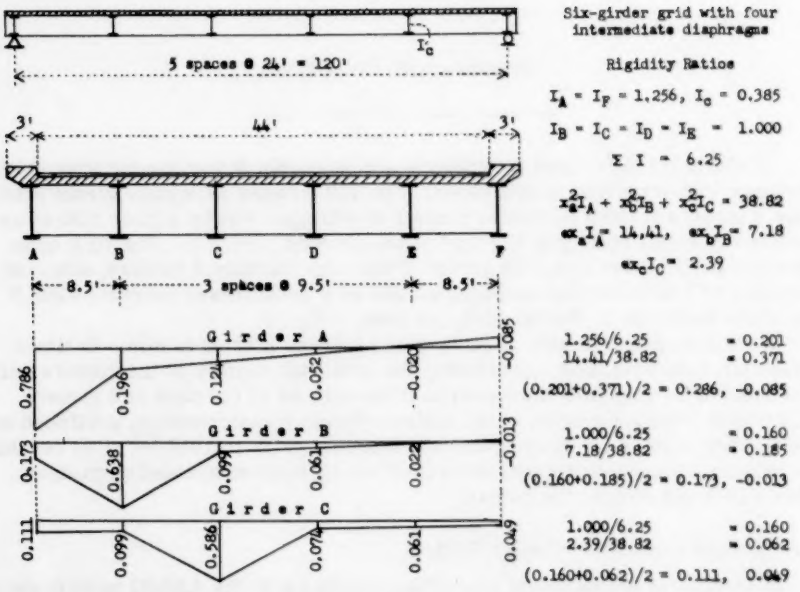

FIG.1 - 120 Ft Span Plate Girder Bridge. Empirical Influence Lines of Load Distribution

TABLE 1 - 120 Ft Span Bridge. Moments due to H 20-S 16-44 Truck Loading - 30% Impact, KFt

Method / Girder	Author	Empirical Influence Lines		AASHO Specifications 6th Edition	7th Edition
A	1,450	1,433	1,311*	828*	1,699
B	1,720	1,783	1,763*	2,203	2,003
C	1,640	1,629	1,629	2,326	2,114

*Center line of first wheel is 2.75 ft from Girder A

behavior of a reinforced concrete 4-girder grid model having similar stiffness ratios.[3] Fig. 2 shows the computed load distribution diagrams of this model when the torsional stiffness of the girders was neglected and considered. The latter closely approximates the measured values. A load applied on this model at the relative location of that shown by the author in Fig. 9, p. 1303-22, produced a similar load distribution diagram to that measured on this 70 ft span structure.

Introducing the geometric and rigidity ratios of this 70 ft span bridge, as computed from scaled data, into Eqs. (14), and (15), [4] the measured values were obtained. These greatly deviate from the diagram marked by the author, "Lazarides" in Fig. 9, p. 1303-22, which was obtained by electronic computer. The degree of the deviations is the same as indicated by the model data in Fig. 2. The effect of the torsional grid stiffness on load distribution in the model and in this 70 ft span bridge are summarized in Table 2.

CONCLUSIONS

The accuracy of the data describing the structure must fit the character of the tool of analysis. Neglecting torsional stiffness, when it is evidently significant, renders the electronic computer practically worthless. Measurements on built bridges furnish reliable static and dynamic basic information. The value of the design computations is gaged by their accuracy in assuring the desired structural performance and not by the complexity of the methods and tools used in the analysis.

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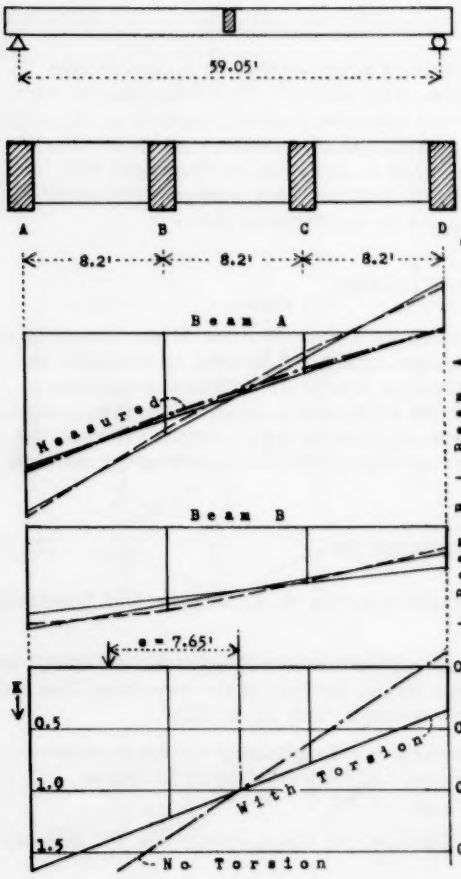
1. "Zur Berechnung von Fundamentrahmen," by W. L. Andrée, Der Eisenbau, Vol. 8, 1917, pp. 219-.
2. "Prestressed Bridge Designed for Crane Load at Niagara River Weir," by A. M. Lount, Discussion by Louis Balog, Journal of the American Concrete Institute, Vol. 28, June 1957, Proceedings, Vol. 53, p. 1238.
3. "Enfluss der Verdrehungssteiffigkeit der Haupttrāger auf die Lastverteilung beim Trāgerrost nach Rechnung und Versuch," by J. Schōttgen, Bautechnik-Archiv, Vol. 1, 1947, pp. 35-46.
4. "Grid bridge design," by Louis Balog, Civil Engineering, Vol. 27, January 1947, p. 42.



Prototype dimensions are shown

Four girders of equal section framed by a center diaphragm

Rigidity Ratios

$I_g = 1$ $I_t^g = 0.505$ $I_c = 0.619$

$z = 30$ $G/E = 0.397$ $\bar{I} = 0.591$

Ordinates of Load Distribution Diagrams

		Computed: No Torsion, I_c = 0.619	Computed: No Torsion, I_c = Infi.	Computed: With Torsion, I_c = 0.619	Computed: With Torsion, I_c = Infi.	Measured
Beam A	A	0.720	0.700	0.534	0.516	0.541
	B	0.379	0.400	0.324	0.339	0.328
	C	0.081	0.100	0.144	0.161	0.140
	D	-0.180	-0.200	-0.002	-0.016	-0.009
Beam B	A	0.379	0.400	0.324	0.339	
	B	0.324	0.300	0.298	0.280	
	C	0.215	0.200	0.234	0.220	
	D	0.082	0.100	0.144	0.161	

$e = 7.15(24.6/23) = 7.65'$

t_A and t_D =

$0.25 \pm 0.3(7.65/8.2)0.591$

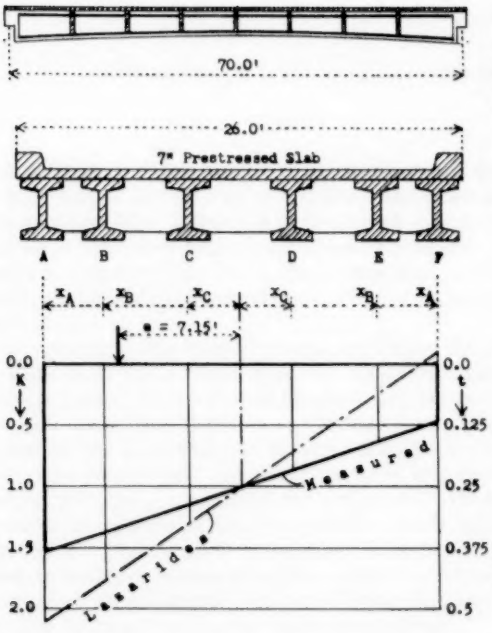
$t_A = 0.415$ $t_D = 0.085$

$\Sigma I_g t_g = t_A + t_B + t_C + t_D = 1.0$

Area K diagram/24.6 = 1.0

$12.3(1.66 + 0.34)/24.6 = 1.0$

FIG.2 - Effect of the Torsional Rigidity of the Girders on Load Distribution in a Grid Model



Six precast prestressed girders framed by seven prestressed intermediate diaphragms with full moment connections. Seven-inch prestressed anchored slab. Depth of girders at midspan 3.5 feet.

Ordinates of Load Distribution t_A and t_F -

$$I_A \left[\frac{1}{\Sigma I_g} \pm \frac{x_A \, e}{\Sigma I_g \, x^2} \cdot \bar{T} \right]$$

The measured ordinates result from this equation if the following values are introduced:

$I_A = 1, \quad I_B = 0.42, \quad I_C = 0.63$

$x_A = 11.5, \quad x_B = 8, \quad x_C = 3$

$\bar{T} = 0.532$

$t_A = 0.38$ $\qquad$ $t_F = 0.11$

$\Sigma I_g t_g = 1(0.38 + 0.11) + 0.42(0.34 + 0.15) + 0.63(0.28 + 0.20) = 1.00$

The area of the K diagram divided by its width must equal unity

$11.5(1.52 + 0.48)/23 = 1.00$

FIG.3 - 70 Ft Span Concrete Girder Bridge. Computed and Measured Load Distribution

TABLE 2 - Effect of the Torsional Grid Stiffness on Load Distribution

Torsional Stiffness	4-Girder Model				70-Ft Span Bridge			
	t_A	t_D	K_A	K_D	t_A	t_F	K_A	K_F
Neglected	0.530	-0.030	2.12	-0.12	0.525	-0.025	2.10	-0.10
Considered	0.415	0.085	1.66	0.34	0.380	0.112	1.52	0.48

PIN-ENDED GABLED FRAMES[a]

Discussions by Thomas C. Coleman and Thomas D. Y. Fok

THOMAS C. COLEMAN,[1] A.M. ASCE.—The author of the paper has directed attention to interesting structural relationships existing in a gabled frame. The relationships discussed in the paper are derived from the same equations that form the basis of such standard procedures as slope deflection and moment distribution. The method proposed in the paper seems to have advantages for a single span frame, but not so much so for frames of more than one span.

In analyzing any indeterminate structure the stresses depend on the elastic properties of the structure as well as on the statical conditions. In the continuous frame, the elastic properties are represented by the equations as given in the paper. Additional conditions are imposed by the necessity of balancing the moments and shears throughout the structure, and in the structure as a whole; also the amount of translation of the joints must be consistent with the geometry of the structure. All of these conditions can be expressed by equations, thus involving a relatively large number of simultaneous equations. The methods of moment distribution and the method under discussion both represent efforts to avoid the necessity of solving the large number of simultaneous equations otherwise involved. The desirability of doing this is usually accepted without challenge.

However, from a mathematical point of view the structure and the simultaneous equations should represent identically the same thing, the equations expressing in symbols what the structure is in reality. The solution to the equations is actually produced by the analysis of the structure. In other words, after translating the structure into mathematical relationships, the conclusion of these relationships is reached by avoiding the mathematical procedures that have been developed for doing that very thing.

The reason for this condition is probably that the solution of simultaneous equations is regarded as too elementary a subject to be discussed by engineers. The subject is consistently taught as a part of elementary algebra, and taught on the basis of being given exact data and requiring exact results; whereas the engineer is usually more interested in cases where approximate data is available, and it is desired to produce results of an accuracy consistent with the accuracy of the data.

When it is required to analyze a structure and the various relationships have been reduced to mathematical terms, it would seem that the very best possible procedure for obtaining the results should be available from proper

a. Proc. Paper 1353, September, 1957, by James Chinn.

1. Asst. Prof., Dept. of Civ. Eng., Northeastern Univ., Boston, Mass.

examination of the mathematical relationships themselves, without being referred back to the structure or other subject matter from which they were derived.

The logic of the above reasoning would seem to indicate that a better method of solving a large number of simultaneous equations should be available for obtaining acceptable approximate solutions, and that the search for such would be justified. Some years ago the same two span gabled frame was discussed in a paper published by the writer(1) from the point of view outlined above, and while the results were hardly conclusive, they were not without promise.

REFERENCE

1. "The Equations of the Continuous Frame" in the Journal of the Boston Society of Civil Engineers, Vol. 40, No. 2, April 1953.

THOMAS D. Y. FOK,[1] A.M. ASCE.—The author is to be commended for presenting a concise and simplified method for solving pin-ended gabled frames of one or two spans. Although several methods of solution for gabled frames are well known, their applications to frames containing members of variable cross-section are cumbersome. Column analogy, for instance, involves approximation by dividing the length of members into many small segments. The ordinary method of consistent deformation, on the other hand, requires a series of integrations along the members of the frame in order to solve the redundancy. By using moment distribution coefficients which are readily available for many variations of cross-section, tedious numerical computation can indeed be avoided.

However, the greatest advantage of the proposed method lies perhaps in the possibility of its extension to the solution of gabled frames whose ends are either fixed or partially fixed. This can be done by superimposing the effects of the moments at the fixed ends to the solution of the pin-ended frame. Hence, the procedures of analysis need be modified only slightly. As an example, a single span gabled frame with both ends fixed has been analyzed by this method and is presented here. The same procedure can be applied to frames with partially fixed ends.

Consider a gabled frame having same dimensions and loading conditions as Example 1 in the paper, except that both ends of the frame are fixed. Hence, the original tabulated properties of members can be used, and only the missing quantities pertaining to the extension are furnished here.

$$\text{Member DA: } K(1\text{-}CC_{opp}) = 6.8(1 - 0.71 \times 0.46) = 4.58$$
$$\text{Member EB: } K(1\text{-}CC_{opp}) = 5.1(1 - 0.66 \times 0.47) = 3.52$$

Since the structure is statically indeterminate to three degrees, its deflected shape can be represented by the superposition of the following:

(a) deflection due to applied loading, horizontal reaction being removed;
(b) deflection due to H_E;
(c) deflection due to M_D; and
(d) deflection due to M_E.

1. Design Engr., Richardson, Gordon & Associates, Pittsburgh, Pa.

These conditions are shown by bases (a), (b), (c) and (d) in Fig. I.

It is obvious that cases (a) and (b) are identical to those of Example 1 in the paper. Where additional coefficients are needed, they are given in Table I. Hence, the angle changes at the ends can be computed. Cases (c) and (d) are presented respectively in separate Tables II and III, which are self-explanatory with reference to Fig. II.

The results of each of these cases have been summarized in Fig. I. The problem can be solved by setting up three simultaneous equations relating H_E, M_D and M_E as follows:

$$
\begin{aligned}
1953 \quad H_E - 32.31\, M_D - 42.97\, M_E &= 1033 \\
32.31 H_E - 1.208 M_D - 0.391 M_E &= 18.66 \\
42.97 H_E - 0.391 M_D - 1.679 M_E &= 16.96
\end{aligned}
$$

from which

$H_E = 0.742$ k; $M_D = 1.74$ k-ft; and $M_E = 8.60$ k-ft.

The above solution has been carried out by slide rule. The accuracy may be improved with the aid of a desk calculator. However, a check of the problem by moment distribution reveals that the results are close enough for practical purposes.

It should also be noted that this method can easily be adapted to digital computer. For different loading conditions, the solution can be simplified considerably by repeating the same procedures.

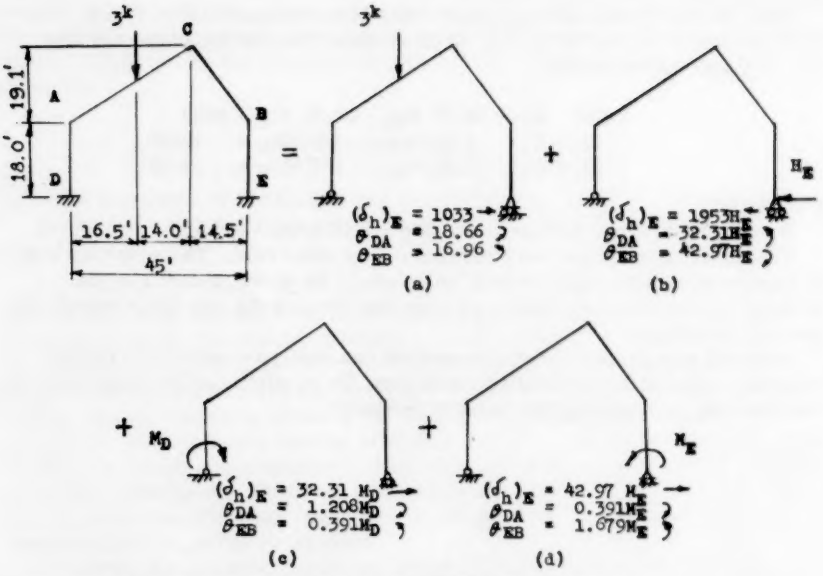


FIG. I SUPERPOSITION BY CONSISTENT DEFORMATION

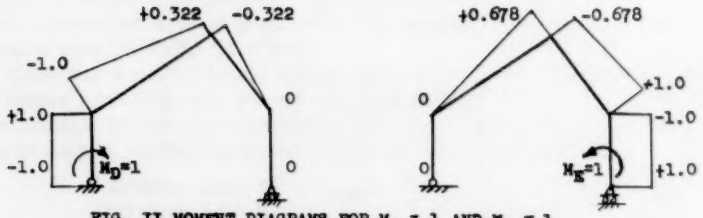


FIG. II MOMENT DIAGRAMS FOR $M_D = 1$ AND $M_E = 1$

TABLE I MISSING QUANTITIES IN CASES (a) & (b)

Member	Case (a)		Case (b)	
	DA	EB	DA	EB
$C_{opp}\Delta M_{opp}$	0	0	-8.28	+8.46
$\Delta M - C_{opp}\Delta M_{opp}$	0	0	+8.28	-8.46
$\frac{\Delta M - C_{opp}\Delta M_{opp}}{K(1 - CC_{opp})}$	0	0	+1.81	-2.40

Case (a)

$$\theta_{DA} = [(45)(12.06) + (14.5)(20.48)]/45 = 18.66$$

$$\theta_{EB} = [(45)(3.08) + (30.5)(20.48)]/45 = 16.96$$

Case (b)

$$\theta_{DA} = 1.81 + 19.12 + (35.31)(14.5)/45 = 32.31$$

$$\theta_{EB} = 2.40 + 16.63 + (35.31)(30.5)/45 = 42.97$$

TABLE II ANGLE CHANGES AND DEFLECTION - CASE (c)

Member	DA	AD	AC	CA	CB	BC	BE	EB
M	-1.000	+1.000	-1.000	+0.322	-0.322	0	0	0
M^F	0	0	0	0	0	0	0	0
$\Delta M = M - M^F$	-1.000	+1.000	-1.000	+0.322	-0.322	0	0	0
$C_{opp}\Delta M_{opp}$	+0.460	-0.710	+0.193	-0.600	0	-0.190	0	0
$\Delta M - C_{opp}\Delta M_{opp}$	-1.460	+1.710	-1.193	+0.922	-0.322	+0.190	0	0
$\frac{\Delta M - C_{opp}\Delta M_{opp}}{K(1 - CC_{opp})}$	-0.319	+0.242	-0.491	+0.380	-0.105	-0.062	0	0
$\Delta\lambda$		+0.733		+0.485		+0.062		

$(\delta_h)_E = (0.733 + 0.062)(18) + (0.485)(37.1) = 32.31$ →

$\theta_{AD} = 0.319 + 0.733 + (0.485)(14.5)/45 = 1.208$)

$\theta_{EB} = 0.062 + (0.485)(30.5)/45 = 0.391$)

TABLE III ANGLE CHANGES AND DEFLECTION - CASE (d)

Member	DA	AD	AC	CA	CB	BC	BE	EB
M	0	0	0	+0.678	-0.678	+1.000	-1.000	+1.000
M^F	0	0	0	0	0	0	0	0
$\Delta M = M - M^F$	0	0	0	+0.678	-0.678	+1.000	-1.000	+1.000
$C_{opp}\Delta M_{opp}$	0	0	+0.407	0	+0.590	-0.400	+0.660	-0.470
$\Delta M - C_{opp}\Delta M_{opp}$	0	0	-0.407	+0.678	-1.268	+1.400	-1.660	+1.470
$\frac{\Delta M - C_{opp}\Delta M_{opp}}{K(1 - CC_{opp})}$	0	0	-0.168	+0.279	-0.414	+0.457	-0.334	+0.418
$\Delta\lambda$			+0.168		+0.693		+0.791	

$(\delta_h)_E = (0.168 + 0.791)(18) + (0.693)(37.1) = 42.97 \rightarrow$

$\theta_{DA} = 0.168 + (0.693)(14.5)/45 = 0.391$ ↻

$\theta_{EB} = 0.418 + 0.791 + (0.693)(30.5)/45 = 1.679$ ↻

DESIGN OF MULTI-LEVEL GUYED TOWERS: WIND LOADING[a]

Discussion by Herbert S. Saffir

HERBERT S. SAFFIR,[1] M. ASCE.—In this paper covering wind loadings on multi-level guyed towers (and in its companion paper on structural analysis) the authors have given proper consideration to the loads acting on a structure and to the intensity and direction of these loads, before proceeding to the structural analysis and subsequent design based on these loads. The writer feels that too often there is blind acceptance of loads and loading conditions which are not always applicable to a particular structure or condition. In too many cases, insufficient thought is given to establishing loads and loading conditions; a highly rigorous structural analysis may be based upon a faulty assumption of loads and loading conditions. Any structure in which wind velocity fixes the most important load should certainly be analyzed and designed only after due consideration has been given to establishing velocity pressures for heights involved, establishing shape or form factors for the members involved, and making investigations of all possible combinations of wind directions, shielding effects, and studies of the structure as a whole where openings are involved.

The writer agrees that a 100 year storm could well form the basis for a wind load choice for structures whose failure might cause loss of life or extensive property damage, and where data is available to establish the magnitude of this storm. Because of lack of sufficient wind velocity records, caution should be used in attempting to extrapolate data from existing records, reaching back only thirty or forty years, for example. Where a design storm frequency greater than 1 in 100 years is used with a corresponding velocity pressure reduction, the insurance rates for wind damage could be varied. However, the writer does not believe this is practical because of the higher labor and material costs of structures of this type, and because insurance rates are generally established for local areas on the basis of the amount of previous overall wind damage in that area.

Estimates of frequency of occurrence of tropical cyclones should be made with extreme care. The lack of observation data useful to structural engineers and the difficulties involved in extrapolating the meager data available and in establishing cyclic changes, make it difficult to set up design criteria and frequencies for this type of storm.

The authors state that "winds of tropical cyclone type over 124 mph should be assumed of constant velocity independent of altitude". The writer would appreciate an explanation of the basis of this statement. Data heretofore available indicates that wind velocities in tropical cyclones (as well as in the

a. Proc. Paper 1355, September, 1957, by Edward Cohen and Henri Perrin.
1. Asst. County Engr., Dade County Eng. Dept., Miami, Fla.

non-tropical storms), increase from the surface to about a 3000 foot elevation. Velocity profiles in tropical cyclones, for these lower layers in which structural engineers are interested, are logarithmic or exponential. For the next 3000 feet, tropical cyclonic winds show little change in velocity; throughout the upper layers of the vortex there is a rapid decrease with elevation, and at heights of 40,000 feet or so the presence of the cyclone often can not even be detected.

Data available from the studies made at Brookhaven National Laboratory on hurricanes Carol and Edna of 1954 indicate a variation of wind velocity with height. Actual wind observations in hurricanes are scarce and fragmentary. In the study made at Brookhaven, the meteorological facility located there was utilized to advantage since hurricanes Carol and Edna passed within 25 and 90 miles distance, respectively, of the facility. Since opportunities for the study of hurricane winds are limited, the writer would appreciate mention of additional data available upon which the authors base their assumption of constant velocity winds in tropical cyclones, regardless of elevation.

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1. Smith, Maynard E. and Singer, Irving, Hurricane Winds at Brookhaven National Laboratory, January, 1956.

2. Myers, Vance A., Interim Report on Characteristics of Hurricanes, National Hurricane Research Project (unpublished).

DESIGN OF MULTI-LEVEL GUYED TOWERS: STRUCTURAL ANALYSIS[a]

Discussions by Kuang-Han Chu, Myle J. Holley and Vitelmo Bertero and I. Earl Lewis, Jr.

KUANG-HAN CHU.[1]—This valuable paper by Messrs. Cohen and Perrin is marred by a lack of definition of certain terms and by errors or misprints in equations, particularly Eq. (24). The following corrections are suggested. In Eq. (14) "Q" should be "Q^2". Eq. (14a) is correct if W is defined as the total load normal to the chord. Then $Q = \frac{W}{2}$, not $\frac{W}{2 \cos \alpha}$ and the term $\frac{Q^2}{3} \frac{\tan^2\alpha}{\cos^2\alpha}$ in Eq. (14b) should be $\frac{Q^2}{3}\tan^2\alpha$. In Eq. (15), the following definitions should be given: h height of tower; V_{th} vertical reaction on tower; A_t total cross-sectional area of tower legs; E_t modulus of elasticity of tower; ϵ_t coefficient of thermal expansion; and Δt° as temperature change in degrees. There are also errors in Eq. (24):

1. In Eq. for G_2, the last term τ should be τ_3.
2. In Eq. for G_3, the term $(4I_o + 2\delta)$ should be $(4I_o - 2\delta)$, and the term $18I_o^2$ reads like $^{1}8I_o^2$.
3. In Eq. for G_4, the first term $-\tau$ should be $-\tau_1$ and the term $(12I_o^3 - 24I_o^2\delta - \delta^3)$ should be $(12I_o^3 - 24I_o^2\delta + \underline{12I_o\delta^2} - \delta^3)$.
4. In Eq. for G_5, the terms of δI_o are not clear. For example, $10\delta I_o^3$ reads like $106I_o^3$. It would be preferable to write the terms as $I_o\delta$.
5. In Eq. for G_6, the term $-\frac{\tau 2}{2} I_o{}^4(\delta - I_o)^2$ should be $-\frac{\tau 2}{2} I_o{}^4\underline{\delta}(\delta - I_o)^2$.
6. The Eq. $\tau_2 = \frac{c}{AE_2}$ should be $\tau_2 = \frac{c}{AE}\sec\alpha$.
7. The Eq. $\tau_3 = \frac{cQ2}{6\cos\alpha}$ should be $\tau_3 = -\frac{cQ_x^2}{12\cos\alpha}$.

In conclusion, it should be noted that the above errors might be due to oversight or misprint. However, they should be corrected at an early date so that this valuable paper can become more useful to the readers. The writer would like to congratulate the authors for their significant contribution to the subject.

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a. Proc. Paper 1356, September, 1957, by Edward Cohen and Henri Perrin.

1. Associate Prof., Dept. of Civil Eng., Illinois Inst. of Technology, Chicago, Ill.

MYLE J. HOLLEY, JR.,[1] A.M. ASCE and VITELMO BERTERO,[2] A.M. ASCE.—By indicating the major factors to be considered in the analysis of multi-guyed towers the authors have made an important contribution to the profession. The paper should be of particular value to engineers concerned for the first time with this type of structure.

The writers had occasion to study the behavior of multi-guyed towers a few years ago. This study culminated in an unpublished report and in a thesis[3] by the junior writer in partial fulfillment of the requirements for the degree of Master of Science in Civil Engineering. This background enhanced the writers' interest in the authors' paper, and provided the basis for a substantial portion of the discussion which follows.

To start with the relatively unimportant, it is suggested that a few of the authors' equations appear to have been misprinted. When the two equations (19) are added (rather than equated as indicated by (20)), the functions $\cos\alpha$ and $\sec\alpha$ cancel out. Thus in Eqs. (24) should not the function $\cos\alpha$ either be deleted from the expressions for $\mathcal{C}_1$ and $\mathcal{C}_3$, or added to the denominator of the expression for $\mathcal{C}_2$? Also in the expressions for G_1 through G_6, Eqs. (24), some of the numerical coefficients and signs appear to be misprints.

While the authors define spring constant, K, as force per unit displacement, their Eq. (26) expresses K as a total force corresponding to total displacement μ. Should not the right side of Eq. (26) be divided by μ?

It should perhaps be pointed out that the computation of coefficients X_{jk} should be based upon beam-column theory; i.e., the modifying influence of axial load should be included if the factor L/j is not small. Not only stiffness and carry-over factors but also the end moments associated with support displacement would be subject to modification.

Quite frequently the tower geometry and anticipated leg stress associated with tower axial load are such as to render the use of beam-column theory unnecessary. If it is recognized that the authors' expression for L/j can be written (for a triangular tower) as

$$\frac{L}{j} = 2.45\left(\frac{L}{a}\right)\sqrt{\frac{f_o}{E}},$$

where, a is the width of each tower face
f_o is the unit stress associated with axial force
E is the modulus of elasticity,

a preliminary estimate of L/j can readily be made. The authors' curves of Fig. 10a, and curves of maximum bending moment augmentation factors vs L/j for typical loadings such as shown in the authors' Fig. 10b, can then be used to determine whether beam-column theory need be applied. Certainly the trend toward high strength alloy steels (with accompanying high values of f_o) would seem to imply that this refinement will be increasingly important, particularly for very large towers. Moreover, if design is based on an ultimate load basis, (and the writers believe this to be highly desirable) the larger values of f_o occurring at ultimate load may well require the application of beam-column theory.

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3. "Investigation of Large Multi-Guyed Towers," by V. Bertero, Mass. Inst. of Technology, Department of Civil and Sanitary Engineering, June 1955.

The authors recommend that the total vertical force applied by the guys at a particular guy level be taken equal to a constant times the horizontal reaction developed by the guy group. The writers agree that it is impractical and unnecessary to use a more precise relationship. In all practical cases the moments of vertical forces produce only very small increments of horizontal reactions. On the other hand, the tower bending moments occasioned by eccentricity of the guy connections, (as given by the authors' Eq. (32)), generally contribute substantially to total tower moments, and always should be considered.

The authors' choice of redundants leads to a set of simultaneous equations involving the support displacements as unknowns. The writers have found it more convenient to use internal moments at the supports as unknowns. Solution for these unknowns by "successive adjustment of internal angle discontinuities" is a rapid process which can be carried out by relatively unskilled personnel. This approach is particularly advantageous when the number of unknowns is large.

The writers investigated the relative merits of large and small values of initial guy tension. Usually there is a minimum acceptable initial guy tension governed by limitations on permissible displacement at the top of the tower. Furthermore, as the initial guy tension is increased the tower leg forces (due to combined axial load and bending) also increase. Thus for towers designed to a "permissible working stress" increased initial guy tensions require heavier leg sections. There are, however, other factors bearing on the question of optimum initial guy tensions. First, it can be shown that the effects of guy tension maladjustments (due either to improper installation or to service conditions) are much more likely to cause excessive tower bending stresses if the guy array, as a whole, is designed for low initial tensions. This is physically apparent since a "soft" array of guys involves large displacements at each guy level; if one guy group is too stiff or too flexible, undesirable relative support displacements, which also are large, can result. These relative displacements are accompanied by significant changes in tower bending moments from the values contemplated in the design. Second, total tower leg forces (due to combined axial load and bending) are much less sensitive to initial guy tensions at ultimate load than at "working load". Thus in a tower designed on the working stress basis the use of heavier leg sections, occasioned by higher initial guy tensions, results in a larger ultimate load capacity. Thereby a design which is less sensitive to guy tension maladjustments is required to have the greater factor of safety. This is not logical. It appears that the question of optimum initial guy tensions cannot be satisfactorily resolved unless design is based on ultimate strength.

The authors' statement, "The actual factor of safety may be computed in terms of load factors applied to the wind and dead load", implies a recommendation for ultimate load design. Depending upon the magnitude of wind load factor employed, the design wind velocity might have either its present significance (wind velocity at which tower stresses approximate typical permissible working values) or wind velocity below which the tower will not collapse. The writers believe a general adoption of the latter concept is very much to be desired. Regardless of the implied significance of "design wind conditions", however, the writers feel strongly that analysis should be performed for ultimate conditions; it is inferred that the authors likewise hold this view. The ratio of critical tower leg stress to permissible working stress may differ greatly from the ratio of load intensities producing these

stress levels. Thus design on a working stress basis not only obscures the factor of safety, but may invalidate studies of the influence of initial guy tension and other variables.

The authors apparently endorse the widely held opinion that design for co-linear support displacements is desirable. Adherence to this concept results in tower bending moments which are substantially greater at points of support than in the midspan zones. As was pointed out by the authors in their companion paper on Wind Loading, the wind direction producing maximum values in the shaft of a triangular tower is normal to a face. For this wind loading the difference between leg forces at supports and at midspan is further accentuated by the fact that only one compression leg resists moment at the former location whereas two compression legs share the moment at midspan. It is not uncommon to locate transitions in leg areas short distances above support points and to maintain constant leg sections over most of the length of each span. Thus the leg section used is excessive everywhere except at its lower end and where it crosses a support.

By departing from the principle of co-linear support displacements it is possible to alter the tower bending moment diagram, decreasing the peaks which occur at support points and increasing the midspan values. This permits full utilization of the leg section at midspan and at the supports, rather than at the latter points only, and permits a smaller leg section due to the reduced support point moments.

An interesting aspect of design for "balanced tower moments" is that this approach is very much more direct than design for co-linear support displacements. (a) Equations of static equilibrium yield the values of support reaction required to produce the desired moments. (b) The uppermost guys are designed to provide their reaction at an acceptable displacement. (c) Displacements of intermediate support points from the overall chord are computed from the known bending moment diagram and added to the chord displacements. (d) Intermediate guys are designed to provide the required reactions at the required displacements. This process would be carried out for the apparent critical wind condition; it must be acknowledged that conventional analyses might still be required to verify adequacy of the design for other wind conditions.

The "balanced tower moments" concept is, of course, applicable either in design for working stresses or in design for ultimate loading. For reasons already cited the writers prefer the latter.

Adjustment of guy tensions to provide the proper stiffnesses in accordance with "balanced tower moments" should be no more difficult than adjustment to provide co-linear support displacements. Furthermore, the adverse effects of maladjustment of guy tensions do not appear to be any more serious in one case than in the other. In view of potential material economies which may be achieved through "balanced tower moments" the writers suggest that this design concept merits serious consideration.

It is hoped that none of the foregoing discussion will be interpreted as unwarranted digression, but rather that it will be deemed to have a useful relationship to the authors' excellent presentation.

I. EARL LEWIS, JR.,[1] A.M. ASCE.—The authors are to be congratulated

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for the excellent and complete treatment of this subject. While several designers have been employing similar analyses, it is probably true that there are in existence some structures which have not had the benefit of such analysis, but which, due to their tremendous height alone, must be considered important to the engineering profession and to the general public. Here a design procedure has finally been published which can be openly discussed with the ultimate hope that all future important structures of this type will be uniformly and accurately analyzed.

The following comments in the discussion are of a relatively minor nature and are stated in order to help make the paper even more complete.

In step 4(b) outlined on page 2, the authors state that co-linear deflection at all guy levels is desirable. This statement is correct for a pivoted base tower which is the most desirable design. But it might be well to add the statement that if the base is fixed, many other factors must be given consideration in the design analysis than appear in this paper.

Referring to the statement on page 8 that the tension in the formulas is at the middle point, the following facts could be mentioned for clarification. Since the initial tension is usually measured at the guy anchor, the component of half the total guy weight must be added to this measured value before using in the formulas as initial tension. Also, this same guy weight must be added to the final formula tension to arrive at the maximum guy tension and hence the guy factor of safety.

The comment on page 9 that the leeward guy follows Hooke's law more closely should not be expanded outside of that particular paragraph concerning load applied on the guy. In the overall design with wind on the tower and higher tension in the windward guy, the reverse is true.

Instead of using the wind velocity at the elevation of the center point of each guy as stated on page 9, the approximation would be slightly closer if the average wind velocity (considering the proportionate length in each pressure zone) of each guy was used.

Eq. (14) does not account for the different "Q" values. The value of "Q" at the initial tension consists only of the weight component, while the value at the design load or at the adjusted initial tension referred to on page 13 (c^3) consists of the algebraic addition of the wind component.

Eqs. (25) and Fig. 8 do not allow for the wind load along the length of the guy. The amount of error would depend on the wind direction, wind pressure, and guy length.

Another form of Eq. (23) involving a trial and error solution but containing lower powers of the tensions might prove more advantageous than the form shown. The ease of solution by both forms of the equation could be compared for the initial design computations and also for design check computations.

Sometimes it is desirable to maintain the portions of "S" and of "Q" due to the design wind blowing on the guys constant when varying the horizontal force applied at the guy level. In this case, the values for Fig. 9 would be slightly changed.

While it need not complicate the simultaneous equations or the moment distribution coefficients, the true vertical angle and the true tension of the guys at the upper connection should be used when computing the final vertical component of the guy forces into the tower shaft.

UNIVERSITY RESEARCH IN STRUCTURAL ENGINEERING[a]

Discussions by Bruce G. Johnston and Paul J. Brennan

BRUCE G. JOHNSTON.[1]—The author has prepared a comprehensive survey of the attitudes toward and categories of structural research as practiced in engineering colleges. The writer would have welcomed additional remarks regarding proper ethical standards of university research practice.

Perhaps the author simply took for granted that research at any great university would attain high standards that would warrant its definition as "an objective search for truth." Research in any field, to be worthy of the name, should be uncompromisingly just that.

It is inevitable that worthy research projects will be sponsored by trade associations which have the promotion of a certain product or class of products as a major aim. While such research cannot be called "pure" in the highest sense, it can be and usually is fully objective in presenting all of the ascertained facts and thus promoting the general status of knowledge in the engineering profession. It is also important that the planning of such projects be such as will not give a biased slant to the conclusions.

This is a day when college professors are told the welcome news that they are neither appreciated nor paid enough. Every engineering college professor who is worthy of such increased recognition should look carefully to initial research contracts and obtain prior agreement that "the whole truth, and nothing but the truth" shall be the outcome of his project and that the truths so learned will receive general and unrestricted dissemination.

PAUL J. BRENNAN,[2] A.M. ASCE.—Research is an important activity that has always been a part of university functions. In recent years this activity has received considerable publicity and external financial support. The selection of sponsored projects to be included in a research program is an important discussion.

Research in a university should be of such a nature that

1. The general knowledge of the field is increased and university personnel responsible for this research develop intellectually from it.
2. This intellectual development and knowledge is reflected in the classroom and the student is thus broadened.
3. The university's prestige is enhanced by scholarly publications resulting from such research.
4. Graduate students can partake of research and thus become educated in

a. Proc. Paper 1357, September, 1957, by Frank Baron.

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2. Prof. of Civ. Eng., Univ. of Delaware, Newark, Del.

research techniques and evaluation.

5. If this research is sponsored, a portion of faculty or graduate student salaries can be realized. This can extend the breadth of university course offerings.

University research must be true research that can determine basic knowledge, must be concerned with important matters and should be of such a nature that the interest of the investigator is maintained. It should be within limitations of the facilities of the institution, the time of the principal investigator, and the financial support of the sponsor. This is a practical problem since results are always important to both the researcher and the sponsor and underestimating the scope of the project can lead to inconclusive information. If much work of this nature is attempted, the university loses prestige.

Many sponsors have problems of a specific nature that require immediate answers but are not true research problems. There is no question that these immediate needs are very important and should be studied and, if possible, answered. A publically supported university has a serious obligation to industry within the community served by this university, the state and other departments of the state, and to its citizens, and university personnel should be available to provide consulting services for these problems.

This consulting service can parallel the research program but should be conducted differently from research. Faculty members would study the problem, ascertain what is known about the problem, either from their own experience or through a search of the literature, put limits on the action involved, make other studies as necessary, and then provide the sponsor with definite engineering solutions. The sponsor would obtain these answers in a quick efficient manner and the large amount of report writing and necessary formal procedures that go with research would be eliminated in this service.

The author is to be commended for his clear presentation and evaluation of research in structural engineering.

TORSION ANALYSIS FOR SUSPENSION BRIDGES[a]

Discussion by Kuang-Han Chu

KUANG-HAN CHU.[1]—This interesting and valuable paper by Mr. Sih suffers a great deal due to the fact that notations are not defined at all and that diagrams do not clarify the meaning that the authors intends to convey. It would help the readers not familiar with this subject to define the following notations:

A_h = Area of the diagonals in one panel of the horizontal truss.

A_v = Area of the diagonals in one panel of the Vertical truss.

A_c = Area of one cable.

A_o = Area of one chord of the stiffening truss.

S = Axial stress (force) in member (shearing force resisting torsional moment.)

ω = Wraping (deformation of the chord in axial direction).

δ = A length perpendicular to ω or length along a curve.

E = Modulus of elasticity.

T_x = Torsional moment in a panel.

t_x = Torsional moment per linear foot.

H'_ω = Horizontal component in the cable due to dead load (H_ω) plus Live load (H) as if the bridge were loaded symmetrically with respect to the longitudinal center line of the bridge.

ΔH = Increment in horizontal component in the cable due to torsional moment.

l = Span length (main or side span) or length of a member.

L_S = $\Sigma \int_o^l \frac{d\delta^3}{dx^2}$ for the cable.

y = Vertical ordinates of the cable from the chord.

η = Vertical deflection of the cable.

m = Bending moment per linear foot.

V = Vertical shear.

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a. Proc. Paper 1431, November, 1957, by Nan-Sze Sih.

1. Associate Prof., Dept. of Civil Eng., Illinois Inst. of Technology, Chicago, Ill.

Besides in Eqs. (4), the Eqs. for η and ψ are misplaced. They should be corrected as follows:

For the horizontal system $\psi = \frac{d}{2}\phi$; $\frac{\partial \omega}{\partial s} = \frac{2\omega}{b}$

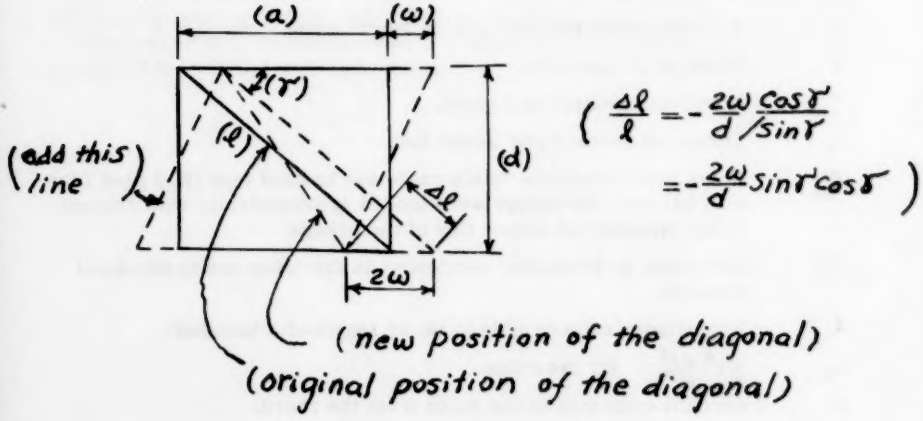
For the vertical system $\eta = \frac{b}{2}\phi$; $\frac{\partial \omega}{\partial s} = -\frac{2\omega}{d}$

In Fig. 2, the width of the horizontal truss should be denoted as "b" and the depth of the vertical truss should be denoted as "d". The deformation of the chord length "a" should be denoted as "ω" in the figure.

In Fig. 4, the vertical forces shown as H'_{ω} etc. should be indicated as vertical shear V due to H'_{ω} etc.

In Fig. 3, the left figure of Fig. 3, the depth of the panel should be denoted as "d". The right figure should be redrawn as shown below. Notes in parenthesis are writer's.

In conclusion, the writer should like to point out that the purpose of the above notes is to make certain that this valuable paper become more intelligible to readers who are not familiar with the subject. He would like to congratulate the author for this worthy contribution.



A SIMPLIFIED DESIGN OF COMPOSITE BRIDGE STRINGERS[a]

Discussion by Paul Spindel

PAUL SPINDEL,[1] J.M. ASCE.—A more exact and perhaps an easier method for choosing the section of steel required for design of a composite stringer exists. It requires as a prerequisite a tabulation of section properties of a number of wind flange sections with cover plates varying in thickness by 1/4 inch increments. Straight line interpolation may be used to get values of sections with other cover plate thicknesses. A sample of a typical tabulation is presented at the end of this discussion. The section properties are computed for n = 10 and n = ∞ (steel section acting alone).

If we use the notation presented in Mr. Hacker's paper, the following equation for the Section Modulous of steel may be derived:

$$f_{DL} + f_{LL+I} = f_s^b$$

Where f_s^b = Steel Stress in Bottom Fiber

$$f_s^b = \frac{M_{LL+I}}{S_{cb}} + \frac{M_{DL}}{S_{sb}} \qquad \langle \times S_{sb}$$

$$S_{sb} f_s^b = \frac{S_{sb}}{S_{cb}} M_{LL+I} + M_{DL} \qquad \langle \div M_{DL}$$

$$\text{Let } \frac{S_{sb}}{S_{cb}} = u$$

$$\frac{S_{sb} f_s^b}{M_{DL}} = 1 + \frac{M_{LL+I}}{u\, M_{DL}}$$

$$S_{sb} = \frac{M_{DL}}{f_s^b} \left[1 + \frac{M_{LL+I}}{u\, M_{DL}} \right]$$

a. Proc. Paper 1432, November, 1957, by Jack C. Hacker.

1. Structural Designer, Brown & Blauvelt, Cons. Engrs., New York, N. Y.

The u value of sections are indicated in the tabulation of sections.

To choose a section for design, the dead load moment and live load moment (including impact) are first found. A u value is assumed. The values of M_{DL}, M_{LL} + I and u are substituted in the formula and a required section is obtained. If the u value of the section differs greatly from the one assumed, a second trial should be made. Since the change in u is relatively small with respect to the change in section, an experienced designer can pick a u consistent with the WF section he will use.

Representative values of u for a 7 inch slab and 7 foot - 0 inch stringer spacing are:

WF	Cov. Pl.	S_s^b	u approx.
36WF300	18x1-1/2	1810	1.25
36WF150	14x3/4	782	1.35
33WF130	14x1/2	579	1.40
27WF94	12x1/2	364	1.45

He may then pick a u value consistent with the final WF which will be used in the design.

The following illustrates the method:

Span: 60'

Stringer Spacing: 7' - 0"

Slab Thickness: 7"

Load: H20-S16-44

Dead Load

Slab =	.612 k/ft.
Steel: Assume	.150
Misc. Details:	.020
	.782 k/ft.

$n = \infty \quad M_{DL} = 352^{1-k}$

$n = 10 \quad M_{LL+I} = 718$

assume u = 1.40

$$S_s^b = \frac{352 \times 12}{18} \left[1 + \frac{718}{354 \times 1.40} \right] = 578 \text{ in}^3$$

Try 33WF130

14 x 1/2 Cov. Pl.

Analysis of this section gives the following stresses:

$f_s^b = 18.10$ ksi $\quad f_s^t = 12.17$ ksi $\quad f_c^t = .54$ ksi

When in addition to the normal Dead Load and Live Load there exists a

Superimposed Dead Load, an approximate solution similar to the one just outlined may be developed.

If the general office practice of using n = 30 for computation of stresses due to Superimposed Dead Load is used, the final stress in the bottom fiber of a section may be expressed by:

$$f_s^b = f_{DL} + f_{LL+I} + f_{SDL}$$

$$f_s^b = \frac{M_{DL}}{S_{sb}^{n=\infty}} + \frac{M_{LL+I}}{S_{cb}^{n=10}} + \frac{M_{SDL}}{S_{cb}^{n=30}}$$

Study of numerous section properties used in common highway bridges i.e. 6' to 10' stringer spacing, 6" to 8" slab has lead to the following approximation:

$$S_{sb}^{n=10} = 1.1\, S_{cb}^{n=30}$$ Therefore

$$\frac{M_{SDL}}{S_{cb}^{n=30}} = \frac{M_{SDL}}{S_{cb}^{n=10}/1.1} = 1.1\,\frac{M_{SDL}}{S_{cb}^{n=10}}$$

and

$$f_s^b = \frac{M_{DL}}{S_{sb}^{n=\infty}} + \frac{M_{LL+I} + 1.1 M_{SDL}}{S_{cb}^{n=10}}$$

The form of the equation for getting the final design sections now becomes

$$S_{sb} = \frac{M_{DL}}{f_s^b}\left[1 + \frac{M_{LL+I} + 1.1 M_{SDL}}{u\, M_{DL}}\right]$$

In the example just solved, a 4" Wearing Surface is to be added. The solution now becomes:

n = ∞	M_{DL} = 352 fk
n = 10	$M_{LL} + I$ = 718 fk
n = 30	M_{SDL} = 111 fk

Assume u = 1.39

$$S_s^b = \frac{352 \times 12}{18}\left[1 + \frac{718 + 1.1 \times 111}{1.39 \times 352}\right] = 640$$

Try 33WF130

This section gives the following stresses:

$$f_s^b = 18.00 \text{ ksi} \quad f_s^t = 13.67 \text{ ksi} \quad f_c^t = .567 \text{ ksi}$$

Sample Tabulation

Section	n=∞			n= 10			u
	I	S_s^b	S_s^t	I_c	S_c^b	S_c^t	
33WF130 PL							
14x1	9,694	745	459	26,720	1,026	3,296	1.38
14x3/4	9,060	663	449	24,260	912	3,351	1.38
14x1/2	8,369	579	437	21,730	797	3,422	1.38
-	6,699	405	405	16,257	566	3,695	1.40

The above Section Properties are based on a 7" x 84" Slab with no haunch.

THE DESIGN OF RIGID FRAME BENTS[a]

Discussions by James M. Gere, Joseph J. Hromadik, D. A. Mackenzie and A. A. Eremin

JAMES M. GERE,[1] A.M. ASCE.—In this paper, Mr. Zimmermann analyzes a rigid frame bent having constant EI by three well-known methods: moment distribution, column analogy, and slope-deflection. The three forms of solution presented by the author are familiar to graduate and undergraduate students taking required courses in statically indeterminate structures in our colleges and universities. All of the material presented by the author can be found in any of a dozen commonly used textbooks. As a matter of fact, the same rigid frame bent which the author analyzes is presented as a sample problem by Grinter.[2] In Grinter's text (pages 98-100) this same rigid frame bent (and with the same loading) is solved in detail, using the slope-deflection method, one of the three methods used by the author.

Several minor errors and omissions have been noted as follows. In Fig. 1, the distance AB (= 10 ft) was omitted. In Fig. 2, line 3, the value 1/2 (1425) should be 712 or 713, but not 714. In Eq. (11), the author finds M_m = 3,142 ft. lb. and then records this result in Fig. 6 as 3137 ft. lb. In Eq. (10a), the quantity V_b should be V_d (this change should be made in four places). In the title of the paper, the word "design" should be changed to "analysis."

JOSEPH J. HROMADIK.[3]—The author has presented three classical methods which have had great appeal to the profession at one time or another. Other methods also have shared in limited popularity. It must be recognized that there is no method which is best for all problems. What is best for one type of problem may be a poor third or fourth for another. It must be emphasized therefore, that it is paramount to the designer to have a working knowledge of several methods in order to make a wise choice. For greatest economy, it is the problem that dictates the selection of a method, not the designer. While in agreement with the author's selection of three useful methods, this writer believes that perhaps a method more suitable to the single framed bent problem would be that of the principles of moment area.

The slope deflection method is one of the older classical methods, and for many problems, is the fastest. Its usefulness is restricted, however, to

a. Proc. Paper 1434, November, 1957, by Richard Z. Zimmerman, Jr.
1. Associate Prof. of Civ. Eng., Stanford Univ., Stanford, Calif.
2. Grinter, L. E., "Theory of Modern Steel Structures," Vol. II, The Macmillan Co., New York, 1949.
3. Senior Project Engr., U. S. Naval Civ. Eng. Research and Evaluation Lab., Port Hueneme, Calif.

conditions of constant section within any given span. It can be used to advantage for multi-framed structures, as the simultaneous equations can be reduced to a computer system.

The moment distribution method has gained prominence because of its adaptability to structures with variable cross-sections. It is an excellent tool for the more complicated structures. It is perhaps in use by more engineers today than any other method. It does have the drawback of requiring repititive computations for various load patterns, and becomes complicated for structures with sloping members.

The column analogy method is most useful as a collateral tool for moment distribution. This method can be most efficiently used to determine properties (stiffness, carry-over, etc.) of variable sections for the moment distribution approach. The column analogy method can be also used to advantage for the solution of single-framed bents.

Application of the moment area principles is an excellent method for the solution of frames and arches. The technique is readily applicable to sloping or curved members. The method allows visualization of elastic behavior of the structure and avoids repeating many computations for various load combinations. It does have the drawback of becoming tedious when the simultaneous equations exceed five, as with the slope deflection method.

It is the purpose of this discussion to add to the author's example a solution by these principles.

The principles: (Reference: Most any textbook on Indeterminate Analysis)

I. The total angle change between any two points, A & B, is the summation of the M/EI area between the two points, or $\sum_{A}^{B}$ Mds/EI.

II. The total vertical departure of point B from the tangent at point A is equal to the moment of the M/EI area, between the points A & B, about point B, or $\sum_{A}^{B}$ Mxds/EI, where x is the horizontal lever arm distance from B to the gravity center of the M/EI area. The horizontal departure is equal to $\sum_{A}^{B}$ Myds/EI, where y is the vertical lever arm.

Method of Solution:

A. Reduce the problem to a determinate, as shown in Fig. 1. The sense of m, V, and H was selected to maintain tension on the outside of the frame, compatible with the superimposed load. Sense of all angle changes and departures is the same, and considered as positive.

B. Compute the angle changes between A & D; compute the departures of D from the tangent at A, as shown in Fig. 2A. The term EI is relative. For the problem the term is constant throughout and taken as one (1). When EI is not constant, the actual or relative values must be included in the computations. For simplification treat each load, including the unknown loads, separately in tabular form.

C. Set up the simultaneous equations for the solution of the unknowns, m, V, and H. To satisfy the original conditions of the problem, the angle change and departures must be equal to zero. The three equations are presented in Fig. 2B.

D. Solve the equations. To illustrate the technique of solving the bent for various load patterns, a general solution for V, H, and m is presented. The

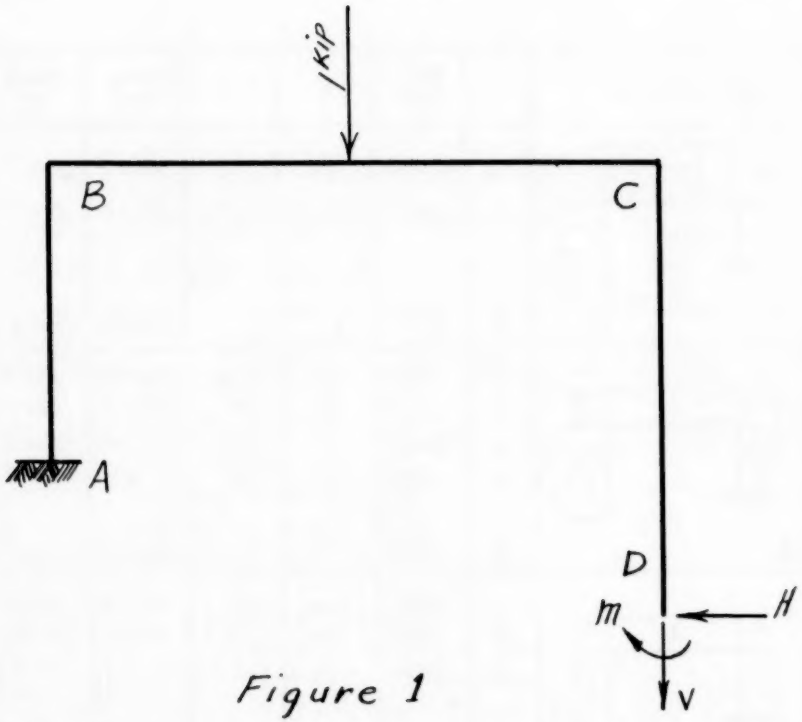


Figure 1

general solution includes the M/EI diagrams of I, II, and III of Fig. 2A and constants a, b, and c for Eqs. (1), (2), and (3) respectively.

It can be noted from the general solution that the M/EI area and moment computations involving V, H, and m (of I, II, and III) are not effected with a change in the load pattern, only the constants a, b, and c change. The general solution therefore applies to any load pattern or combination, whether it be the result of superimposed loads or temperature effects.

The solution for V, H, and m of Load A (Author's example) is given in Fig. 2B. The problem thereafter becomes a determinate. As an example, from Fig. 2A, the moment at connection B is $20V + 15H + m + 10$. To solve the bent for any other load, it is necessary only to determine the constants, which are a function of the simple beam M/EI diagrams for the cantilevered bent as shown in Fig. 1. For temperature changes the constants are the determined departures at D due to changes in member lengths times the actual EI value (in units of kips and feet).

It is the writer's opinion that the moment area concept cannot be excluded from consideration along with other methods of solution for selecting the proper approach to a problem.

D. A. MACKENZIE,[1] M. ASCE.—The author's presentation gives a good idea of the amount of computation involved for each of three methods of

1. Senior Engr., Niagara Mohawk Power Corp., 300 Electric Bldg., Buffalo 3, N. Y.

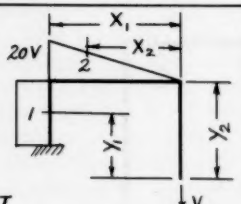
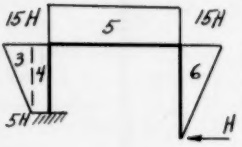
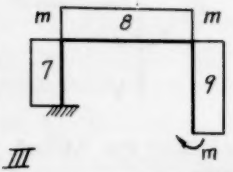
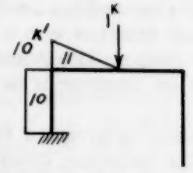
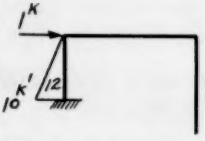
$\frac{M}{EI}$ Diagram		$\frac{Mds}{EI}$	x	y	$\frac{Mxds}{EI}$	$\frac{Myds}{EI}$
 I	1	200V	20	10	4000V	2000V
	2	200V	13.33	15	2667V	3000V
	Σ	400V			6667V	5000V
 II	3	50H	20	11.67	1000H	583H
	4	50H	20	10	1000H	500H
	5	300H	10	15	3000H	4500H
	6	112H	0	10	0	1120H
	Σ	512H			5000H	6703H
 III	7	10m	20	10	200m	100m
	8	20m	10	15	200m	300m
	9	15m	0	7.5	0	112m
	Σ	45m			400m	512m
 IV Load A	10	100	20	10	2000	1000
	11	50	16.67	15	834	750
	Σ	150			2834	1750
 V Load B	12	50	20	8.33	1000	416
	Σ	50			1000	416

Figure 2A

The Equations:

1 $\theta_D = \sum_A^D \frac{M\,ds}{EI} = 0 = 45m + 512H + 400V + a$

2 $\Delta_D^V = \sum_A^D \frac{M \times ds}{EI} = 0 = 400m + 5000H + 6670V + b$

3 $\Delta_D^H = \sum_A^D \frac{M y\, ds}{EI} = 0 = 512m + 6700H + 5000V + c$

Eq.	m	H	V	= Constant
1	45	512	4000	$-a$
2	400	5000	6670	$-b$
3	512	6700	5000	$-c$
1'	1.00	11.38	8.89	$-a/45$
2'	1.00	12.50	16.67	$-b/400$
3'	1.00	13.09	9.77	$-c/512$
4=1'-2'		-1.12	-7.78	$-a/45 + b/400$
5=3'-2'		0.59	-6.90	$-c/512 + b/400$
4'		-1.00	-6.95	$-a/50.4 + b/448$
5'		1.00	-11.69	$b/236 - c/302$
6=4'+5'			-18.64	$(b/448 - a/50.4) + (b/236 - c/302)$
V→			1.00	$(1.065a + 0.1776c - 0.347b) \times 10^{-3}$
H→		1.00		$-6.95V + a/50.4 - b/448$
m→	1.00			$-11.38H - 8.89V - c/45$

NOTE: The constants in this table are to the right of the equal sign.

For Load A

$a = 150 \quad b = 2830 \quad c = 1750$

$V = [1.065(150) + 0.1776(1750) - 0.347(2830)]\,10^{-3} = -0.511^{K}$ ∴ opposite Sense

$H = -6.95(-0.511) - 150/50.4 + 2830/448 = 0.22^{K}$

$m = -11.38(0.22) - 8.89(-0.511) - 150/45 = -1.29^{K'}$ ∴ opposite Sense

To solve Load B (or any other load pattern) use the above V, H, and m relations.

Figure 2B

analyzing rigid frames.

A modification of the method of moment distribution, viz: the method of direct moment distribution, is favored by some engineers and an analysis of the problem by this method is given in this discussion as a matter of interest and comparison.

The method has been presented in various forms by a number of writers, notably L. Y. Lin.[2] It may be regarded as a combination of the methods of fixed points and moment distribution.

To establish the general nomenclature Fig. 1a is shown. The sign convention used will be that moments tending to turn the joint clockwise will be considered positive.

In Fig. 1a let K_{AB}, K_{BA}, K_{BC}, etc. represent the relative fixed end stiffness factors at the various joints. For uniform moments of inertia these are the same at each end, and may be designated as K_1, K_2 and K_3 for members AB, BC and CD, respectively.

Let S_{AB}, S_{BA}, S_{BC}, etc. represent the modified or framed joint stiffness factors.

The modified stiffness and carry over factors may be found by taking two spans at a time such as spans AB and BC, assuming the modified stiffness of the first span joint BA to be known or previously computed. If end A is fixed S_{BA} becomes $K_{BA} = K_1$.

First apply a unit moment, M_{CB} at C, and, assuming temporarily that end B is fixed, $M_{BC} = M_{CB}X$ fixed end carry over factor. Next release joint B and fix end C. The moment at B splits in proportion to the relative joint stiffnesses of the members. The portion going to BC multiplied by the fixed end carry over factor is deducted from M_{CB} and the remainder becomes M'_{BC}. The modified stiffness factor $= \frac{M'_{CB}}{M_{CB}}$ or $S_{CB} = K_2 \times \frac{M'_{CB}}{M_{CB}}$.

If the frame is composed of both beams and columns, the joint stiffness S_{BA} becomes $\sum S_{BA}$ of the combined beam and columns to the left of Member BC.

The modified carry over factor is $\frac{M_{BA}}{M'_{CB}}$.

The same procedure is followed starting from right end to find the joint stiffness and carry over factors on the right. These factors may also be found by the theorem of three moments.

Except for the modified stiffness and carry over factors the method will be somewhat similar to that of the author for moment distribution with side sway correction. Fig. 2 shows the computations involved.

For members of uniform moments of inertia, and single members on either side of the joint, the ratios may be put into formulas as given below, the subscripts being for joint CB and carry over C to B. They may also be made up in the form of curves for convenience.

For the example, ends A and D are fixed so that $S_{BA} = K_1 = 2.00$ and $S_{CD} = K_3 = 1.33$.

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2. A Direct Method of Moment Distribution, Trans. American Society of Civil Engineers, Vol. 63, 1937, pp. 561-605.

$$\text{Modified Stiffness Factor} = \frac{M'_{CB}}{M_{CB}} = \frac{2S_{BA} + 1.5K_2}{2S_{BA} + 2.0K_2}$$

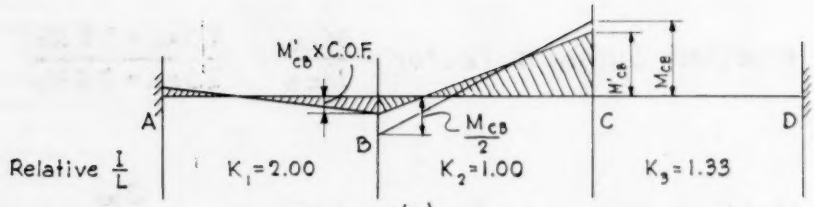
$$\text{Modified Carry over Factor} = COF_{CB} = \frac{S_{BA}}{2S_{BA} + 1.5K_2}$$

$$\frac{M'_{CB}}{M_{CB}} = \frac{2 \times 2.00 + 1.5 \times 1.00}{2 \times 2.00 + 2 \times 1.00} = \frac{5.5}{6} = 0.917$$

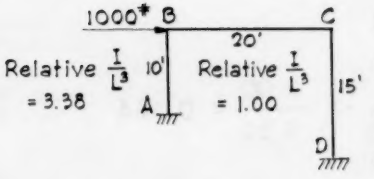
$$COF_{CB} = \frac{2.00}{2 \times 2.00 + 1.5 \times 1.00} = \frac{2.00}{5.50} = 0.364$$

$$\frac{M'_{BC}}{M_{BC}} = \frac{2 \times 1.33 + 1.5 \times 1.00}{2 \times 1.33 + 2 \times 1.00} = \frac{4.17}{4.67} = 0.893$$

$$COF_{BC} = \frac{1.33}{2.67 + 1.5 \times 1.00} = \frac{1.33}{4.17} = .319$$



(a.)



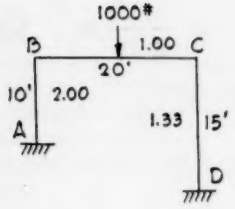
Side Thrust to AB = $\frac{3.38}{4.38} \times 1000 = 772$

Side Thrust to CD = $\frac{1.00}{4.38} \times 1000 = 228$

$FEM_{AB} = 772 \times \frac{10}{2} = 3860'^{*}$

$FEM_{CD} = 228 \times \frac{15}{2} = 1710$

(b.)



$FEM_{BC} = FEM_{CB} = \frac{1}{8} \times 1000 \times 20 = 2500'^{*}$

(c.)

FIG. 1

	A		B		C	D
	$h_1 = 10'$		$L = 20'$		$h_2 = 15'$	
Relative Stiffness = K	$K_1 = 2.00$		$K_2 = 1.00$		$K_3 = 1.33$	
{ Modified Stiffness		2.00	.917 x 1.00 = .917 .893 = .893 x 1.00		1.33	
Bal. Factor		0.691	0.309	.408	.592	
C.O. F.		.50	.319	.364	.50	
Side Thrust of 1000# FIG. 1b						
FEM	+3860'#	+3860			+1710	+1710
Bal. B & C.O.	- 1334	-2668	-1192	- 980	+ 380	+ 190
Bal. C & C.O.	+ 127	+ 254	- 254	- 698	-1012	- 506
Σ	+2653	+1446	-1446	-1078	+1078	+1394
	$H = \frac{4099}{10} = 410$ →		ΣH = 410+165 = 575 Should be 1000# Ratio = $\frac{1000}{575} = 1.74$		$H = \frac{2472}{15} = 165$ →	
Adjusted Σ M	+4616	+2506	-2506	- 1875	+1875	+2426
Vertical Load of 1000#			FIG. 1c			
FEM			+2500	- 2500		
Bal. B & C.O.	- 864	-1728	- 772	- 247	+ 247	+ 123
Bal. C & C.O.	- 186	- 371	+ 371	+1020	+1480	+ 740
	-1050	-2099	+2099	-1727	+1727	+863
	$H = \frac{3149}{10} = 315\#$ ←		Net = 315-173 = 142 Correction = 142 →		$H = \frac{2590}{15} = 173$ →	
Side sway adjustment by proportion from 1000# Thrust						
	+ .142	+.142		- .142		+.142
	x 4616	x 2506		x 1875		x 2426
= Fin. Adjusted Moments	+ 655	+ 356		- 266		+ 344
	- 395	-1743	+1743	-1993	+1993	+1207

FIG. 2

A. A. EREMIN,[1] A.M. ASCE.—It is gratifying that Mr. Zimmermann has shown the procedures of computation of stresses in the rigid frame bent with application of distribution of moments by successive approximation method, column analogy method and method of slope deflection.

In his conclusions Mr. Zimmermann enumerated the characteristic features of the methods in application to the practical numerical example.

With the purpose of checking and valuating the methods of moments computations, illustrated by Mr. Zimmermann, the writer solved the same problem of moments in the two leg bent by the method of graphical distribution of moments.

In Fig. (1) is shown the graphical distribution of moments in the bent with assumption that the bent joints are prevented from the sideway. The bent columns in Fig. (1) are revolved to a horizontal line.

The method of graphical construction of moments in rigid frames is shown somewhere else.(1,2,3)

The elastic points on the beam axis are determined by the graphical construction, known as "three line polygon".

The central moments in BC produced by a given loading of 1000 lbs are:

$$M' = \frac{1000 \times 20}{8} = 2{,}500 \text{ ft. lbs} \qquad (1)$$

The moments produced by the unity sidesway is:

$$M = 6rNa, \qquad (2)$$

where

$$r = \frac{I}{L}$$

$$N = \frac{1}{L(L-a_1-a_2)}$$

a_1 and a_2 are distances to the left and right inflection points, Fig. 1. With Eq. (2) the bending moments produced by the unit sidesway in AB and CD are as follows: $M_{AB} = 6.73$, $M_{BA} = 3.44$, $M_{CD} = 1.84$, $M_{DC} = 3.17$. Distribution of sidesway moments is shown in Fig. 2.

The sidesway moments relating to a loading in Fig. 1 are numerically indicated on Fig. 2.

The final moments corrected graphically for the sidesway effect are shown in Fig. 1.

All graphical constructions were performed on the standard computation sheet 8-1/2 x 11 inches.

It shall be noted that the final moments obtained by the graphics vary with the moments, computed by Mr. Zimmermann only less than five per cent or with proximity which is generally permitted in practice.

It is true as the author stated "a slight error in the signs" in computations by the slope deflection method may entirely change the results.

Likewise, confusion in signs may bring erroneous result in the moments, computed by the method of moment distribution.

In the column analogy method the moment computations are tabulated and

1. Associate Bridge Engr., Calif. State Bridge Dept., Sacramento, Calif.

once the tabulation is checked, the following computations are more or less automatic.

In the graphical construction the signs of moments are regulated by graphical constructions. Therefore, the error in signs may be discovered with a glance on the pictorial construction.

In practice, the design computations should generally be checked by an independent computation. Therefore, designers should be familiar with more than one method of moments distribution. Therefore, the designer should be familiar with at least two methods of computation.

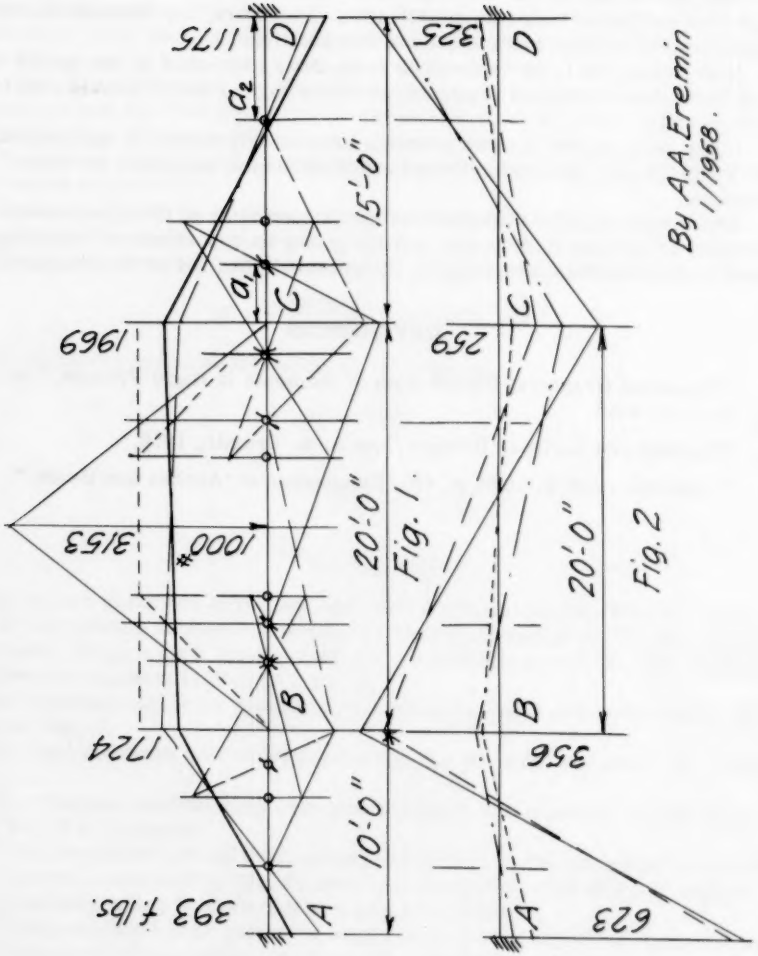
In selecting the favorite methods from those presented by the author the graphical construction of moments, shown in Figs. 1 and 2, should also be considered.

It has been proved that the graphics are not only simple in application, but the graphics give a visual development of moments, adoptable for visual checking.

The author should be credited for the presentation of the most popular methods of moment distribution and for giving an opportunity of selecting a most suitable method according to the personal abilities of the designer.

REFERENCES

1. "Simplified Graphical Distribution of Moments in Rigid Frames," by A. A. Eremin, 1948.
2. "Highway and Railroad Bridges," by A. A. Eremin, 1956.
3. Transaction ASCE, 1950, p. 630, Discussion of "Arches and Bents."


D
C
B
A
a_2
a_1
1175
1969
3153
1000*
1724
393 ft lbs.
15'-0"
20'-0"
10'-0"
325
259
356
623
20'-0"
By A.A. Eremin
1/1958.
Fig. 1.
Fig. 2

ULTIMATE STRENGTH ANALYSIS OF LONG HINGED REINFORCED CONCRETE COLUMNS[a]

Discussion by G. C. Ernst

G. C. ERNST,[1] M. ASCE.—The authors have presented with impressive completeness a long needed theoretical analysis for the ultimate strength of long hinged reinforced concrete columns. Their confirmation of the theory by a detailed comparison with the test results of past investigations is convincing and should lead to the development and use of such charts as represented by Figs. 7 - 10 for ultimate strength design.

The evaluation of the ultimate load for concentrically loaded columns is based soundly upon the stress-strain relationships of the two materials and is defensible for all column lengths. Especially clear is the discussion of the relationship between strain and buckling or material failure. Eqs. (4), (5), (6), and (7) provide a means of solution for the load corresponding to any column length or the critical length for a specific load, for a concentrically loaded hinged column of known cross-section if the stress-strain curves of the steel and concrete are known or assumed. Inasmuch as the solution for Eq. (4) depends upon the end conditions, it is possible to develop the concentrically loaded column curves ($e_2/d = 0$) for any desired end condition.

The extension of the Westergaard-Osgood modification of Karman's method to include unequal end eccentricities, with the graphical solution of Eq. (12) in Fig. 5, is a valuable addition. Also, the discussion of the relationship between strains and buckling or material failure for eccentrically loaded long hinged reinforced concrete columns is helpful.

With regard to the effect of long-time loading, it would seem desirable to test long eccentrically loaded columns under sustained loading in order to confirm or modify the strain assumption made in developing Fig. 10b. This is important, in that it would seem that applications requiring such columns would also frequently involve sustained loads of significant magnitude. Investigation of the effect of end conditions on long eccentrically loaded columns is needed to complete the problem.

The authors are to be commended for the thoroughness of their investigation and the clarity of their presentation.

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a. Proc. Paper 1510, January, 1958, by Bengt Broms and I. M. Viest.

1. Prof. of Civ. Eng. and Director of Eng. Experiment Station, Univ. of Nebraska, Lincoln, Neb.

PROCEEDINGS PAPERS

The technical papers published in the past year are identified by number below. Technical-division sponsorship is indicated by an abbreviation at the end of each Paper Number, the symbols referring to: Air Transport (AT), City Planning (CP), Construction (CO), Engineering Mechanics (EM), Highway (HW), Hydraulics (HY), Irrigation and Drainage (IR), Pipeline (PL), Power (PO), Sanitary Engineering (SA), Soil Mechanics and Foundations (SM), Structural (ST), Surveying and Mapping (SU), and Waterways and Harbors (WW), divisions. Papers sponsored by the Board of Direction are identified by the symbols (BD). For titles and order coupons, refer to the appropriate issue of "Civil Engineering." Beginning with Volume 82 (January 1956) papers were published in Journals of the various Technical Divisions. To locate papers in the Journals, the symbols after the paper numbers are followed by a numeral designating the issue of a particular Journal in which the paper appeared. For example, Paper 1449 is identified as 1449 (HY 6) which indicates that the paper is contained in the sixth issue of the Journal of the Hydraulics Division during 1957.

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MARCH: 1186(ST2), 1187(ST2), 1188(ST2), 1189(ST2), 1190(ST2), 1191(ST2), 1192(ST2)c, 1193 (PL1), 1194(PL1), 1195(PL1).

APRIL: 1196(EM2), 1197(HY2), 1198(HY2), 1199(HY2), 1200(HY2), 1201(HY2), 1202(HY2), 1203 (SA2), 1204(SM2), 1205(SM2), 1206(SM2), 1207(SM2), 1208(WW1), 1209(WW1), 1210(WW1), 1211(WW1), 1212(EM2), 1213(EM2), 1214(EM2), 1215(PO2), 1216(PO2), 1217(PO2), 1218 (SA2), 1219(SA2), 1220(SA2), 1221(SA2), 1222(SA2), 1223(SA2), 1224(SA2), 1225(PO)[c], 1226 (WW1)[c], 1227(SA2)[c], 1228(SM2)[c], 1229(EM2)[c], 1230(HY2)[c].

MAY: 1231(ST3), 1232(ST3), 1233(ST3), 1234(ST3), 1235(IR1), 1236(IR1), 1237(WW2), 1238 (WW2), 1239(WW2), 1240(WW2), 1241(WW2), 1242(WW2), 1243(WW2), 1244(HW2), 1245(HW2), 1246(HW2), 1247(HW2), 1248(WW2), 1249(HW2), 1250(HW2), 1251(WW2), 1252(WW2), 1253 (IR1), 1254(ST3), 1255(ST3), 1256(HW2), 1257(IR1)[c], 1258(HW2)[c], 1259(ST3)[c].

JUNE: 1260(HY3), 1261(HY3), 1262(HY3), 1263(HY3), 1264(HY3), 1265(HY3), 1266(HY3), 1267 (PO3), 1268(PO3), 1269(SA3), 1270(SA3), 1271(SA3), 1272(SA3), 1273(SA3), 1274(SA3), 1275 (SA3), 1276(SA3), 1277(HY3), 1278(HY3), 1279(PL2), 1280(PL2), 1281(PL2), 1282(SA3), 1283 (HY3)[c], 1284(PO3), 1285(PO3), 1286(PO3), 1287(PO3)[c], 1288(SA3)[c].

JULY: 1289(SM3), 1290(EM3), 1291(EM3), 1292(EM3), 1293(EM3), 1294(HW3), 1295(HW3), 1296 (HW3), 1297(HW3), 1298(HW3), 1299(SM3), 1300(SM3), 1301(SM3), 1302(ST4), 1303(ST4), 1304 (ST4), 1305(SU1), 1306(SU1), 1307(SU1), 1308(ST4), 1309(SM3), 1310(SU1)[c], 1311(EM3)[c], 1312 (ST4), 1313(ST4), 1314(ST4), 1315(ST4), 1316(ST4), 1317(ST4), 1318(ST4), 1319(SM3)[c], 1320 (ST4), 1321(ST4), 1322(EM3), 1323(AT1), 1324(AT1), 1325(AT1), 1326(AT1), 1327(AT1), 1328 (AT1)[c], 1329(ST4)[c].

AUGUST: 1330(HY4), 1331(HY4), 1332(HY4), 1333(SA4), 1334(SA4), 1335(SA4), 1336(SA4), 1337 (SA4), 1338(SA4), 1339(CO1), 1340(CO1), 1341(CO1), 1342(CO1), 1343(CO1), 1344(PO4), 1345 (HY4), 1346(PO4)[c], 1347(BD1), 1348(HY4)[c], 1349(SA4)[c], 1350(PO4), 1351(PO4).

SEPTEMBER: 1352(IR2), 1353(ST5), 1354(ST5), 1355(ST5), 1356(ST5), 1357(ST5) 1358(ST5), 1359(IR2), 1360(IR2), 1361(ST5), 1362(IR2), 1363(IR2), 1364(IR2), 1365(WW3), 1366(WW3), 1367(WW3), 1368(WW3), 1369(WW3), 1370(WW3), 1371(HW4), 1372(HW4), 1373(HW4), 1374 (HW4), 1375(PL3), 1376(PL3), 1377(IR2)[c], 1378(HW4)[c], 1379(IR2), 1380(HW4), 1381(WW3)[c], 1382(ST5)[c], 1383(PL3)[c], 1384(IR2), 1385(HW4), 1386(HW4).

OCTOBER: 1387(CP2), 1388(CP2), 1389(EM4), 1390(EM4), 1391(HY5), 1392(HY5), 1393(HY5), 1394(HY5), 1395(HY5), 1396(PO5), 1397(PO5), 1398(PO5), 1399(EM4), 1400(SA5), 1401(HY5), 1402(HY5), 1403(HY5), 1404(HY5), 1405(HY5), 1406(HY5), 1407(SA5), 1408(SA5), 1409(SA5), 1410(SA5), 1411(SA5), 1412(EM4), 1413(EM4), 1414(PO5), 1415(EM4)[c], 1416(PO5)[c], 1417 (HY5)[c], 1418(EM4), 1419(PO5), 1420(PO5), 1421(PO5), 1422(SA5)[c], 1423(SA5), 1424(EM4), 1425(CP2).

NOVEMBER: 1426(SM4), 1427(SM4), 1428(SM4), 1429(SM4), 1430(SM4)[c], 1431(ST6), 1432(ST6), 1433(ST6), 1434(ST6), 1435(ST6), 1436(ST6), 1437(ST6), 1438(SM4), 1439(SM4), 1440(ST6), 1441(ST6), 1442(ST6)[c], 1443(SU2), 1444(SU2), 1445(SU2), 1446(SU2), 1447(SU2), 1448(SU2)[c].

DECEMBER: 1449(HY6), 1450(HY6), 1451(HY6), 1452(HY6), 1453(HY6), 1454(HY6), 1455(HY6), 1456(HY6)[c], 1457(PO6), 1458(PO6), 1459(PO6), 1460(PO6)[c], 1461(SA6), 1462(SA6), 1463(SA6), 1464(SA6), 1465(SA6), 1466(SA6)[c], 1467(AT2), 1468(AT2), 1469(AT2), 1470(AT2), 1471(AT2), 1472(AT2), 1473(AT2), 1474(AT2), 1475(AT2), 1476(AT2), 1477(AT2), 1478(AT2), 1479(AT2), 1480(AT2), 1481(AT2), 1482(AT2), 1483(AT2), 1484(AT2), 1485(AT2)[c], 1486(BD2), 1487(BD2), 1488(PO6), 1489(PO6), 1490(BD2), 1491(BD2), 1492(HY6), 1493(BD2).

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FEBRUARY: 1528(HY1), 1529(PO1), 1530(HY1), 1531(HY1), 1532(HY1), 1533(SA1), 1534(SA1), 1535(SM1), 1536(SM1), 1537(SM1), 1538(PO1)[c], 1539(SA1), 1540(SA1), 1541(SA1), 1542(SA1), 1543(SA1), 1544(SM1), 1545(SM1), 1546(SM1), 1547(SM1), 1548(SM1), 1549(SM1), 1550(SM1), 1551(SM1), 1552(SM1), 1553(PO1), 1554(PO1), 1555(PO1), 1556(PO1), 1557(SA1)[c], 1558(HY1)[c], 1559(SM1)[c].

MARCH: 1560(ST2), 1561(ST2), 1562(ST2), 1563(ST2), 1564(ST2), 1565(ST2), 1566(ST2), 1567(ST2), 1568(WW2), 1569(WW2), 1570(WW2), 1571(WW2), 1572(WW2), 1573(WW2), 1574(PL1), 1575(PL1), 1576(ST2)c, 1577(PL1), 1578(PL1)c, 1579(WW2)c.

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c. Discussion of several papers, grouped by divisions.

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